Integrated Intelligent Research (IIR)
International Journal of Computing Algorithm
Volume: 03 Issue: 03 December 2014 Pages: 226-228
ISSN: 2278-2397

Fractal Boundary Value Problems for Integral and Differential Equations with Local Fractional Operators

T. Henson[1], R. Malarkodi[2]
[1,2]PG & Research Department of Mathematics,
St. Joseph's College of Arts &Science (Autonomous), Cuddalore, India.
Email: thenson1967@gmail.com, malar_msc@yahoo.in

Abstract - In this paper, the local fractional decomposition method is applied to investigate the fractal boundary value problems for the Volterra integral equations and heat conduction equations. The accuracy and reliability of the obtained results of explained using examples.

Keywords- Local fractional decomposition method, heat conduction equations, integral equations, boundary value problem.

I. INTRODUCTION

Fractal is a mathematical set that typically displays self-similar patterns. Fractals are usually nowhere differentiable. These fractals are used in many engineering applications such as porous media modelling, nano fluids, fracture mechanics and many other applications in nanoscale. The fractals nature of the objects must be taken into account in various transport phenomena. The local temperature depends on the fractal dimensions for the transport phenomena in the fractals object. To solve the linear and non-linear problems of ordinary, partial differential equation and integral equations, Adomain introduced a method called the decomposition method. Also inorder to investigate local fractal behaviours of differential equations with fractal conditions,a new tool has been designed called the local fractional derivation.The local fractional variational iteration method local fractional decomposition method etc, are the analytical methods used to solve the differential and integral equations with fractional derivative and integral operator.

II. PRELIMINARIES

Local fractional continuity of functions

Definition1. If there is the relation
$|f(x) - f(x_0)| < \varepsilon^{\alpha}$
with $|x - x_0| < \delta$ for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$.
Now $f(x)$ is called local fractional continuous at $x = x_0$, denoted by $\lim_{x \to x_0} f(x) = f(x_0)$.
Then $f(x)$ is called local fractional continuous on the interval (a, b), denoted by $f(x) \in C_{\alpha}(a, b)$.
Local fractional integrals

Definition2. Setting $f(x) \in C_{\alpha}(a, b)$, local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is defined

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha}$$
$$= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^{\alpha},$$

Where
$\Delta t_j = t_{j+1} - t_j, \quad \Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_{j,\ldots}\}$
And

$[t_j, t_{j+1}], j = 0, \ldots, N-1, t_0 = a, t_n = b$, is a partition of the interval $[a, b]$. For more detail of fractal geometrical explanation of local fractional integral, we see

For any $x \in (a, b)$, there exists
${}_aI_b^{(\alpha)} f(x)$,
denoted by
$f(x) \in I_x^{(\alpha)}(a, b)$.

Here, the following results are valid:

1) If $f(x) \in I_x^{(\alpha)}(a, b)$, one deduce to
$f(x) \in C_{\alpha}(a, b)$.

2) If $a = b$, then we have
$${}_aI_b^{(\alpha)} f(x) = 0.$$

3) If $a < b$, then we have

$${}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x)$$

4) If there is the fractal dimension α=0, then we have

$${}_aI_b^{(0)} f(x) = f(x).$$

5) The sine sub-function can be written as

$$\mathrm{Sin}_{\alpha} x^{\alpha} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1+\alpha(2k+1)]},$$
$0 < \alpha < 1.$

6) There is a useful formula
$$\frac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha}(dx)^{\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right).$$

III. WEYL FRACTIONAL DERIVATIVES

When the fixed limit of differentiation x_0 takes on the singular values ∞ or $-\infty$ is used. We obtain expressions which are often called "weyl fractional derivatives."
They are

$$D^{\alpha}_{x-\infty}F(x) = \frac{(-1)^{-\alpha}}{\Gamma(-\alpha)}\int_x^{\infty} F(t)\,(t-x)^{-\alpha-1}dt$$

(where some suitable branch of $(-1)^{-\alpha}$ must be specified)
and

$$D^{\alpha}_{x+\infty}F(x) = \frac{(-1)^{-\alpha}}{\Gamma(-\alpha)}\int_x^{\infty} F(t)\,(t-x)^{-\alpha-1}dt.$$

IV. ANALYTICAL METHOD

For seek of clarity of the explanation, the local fractional decomposition method will be briefly outlined.The local fractional Volterra integral equation is written in the form

$$u(x) = f(x) + \frac{\lambda^n}{\Gamma(1+\alpha)}\int_0^x K(x,t)u(t)(dt)^{\alpha}, \quad (1)$$

and initial condition

$$u_0(x) = f(x). \quad (2)$$

Substituting $u(x) = \sum_{n=0}^{\infty} u_n(x)$ into equation (1) implies

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \frac{\lambda^n}{\Gamma(1+\alpha)}\int_0^x (x,t)\{\sum_{n=0}^{\infty} u_n(x)\}(dt)^{\alpha}. \quad (3)$$

The components $u_0(x), u_1(x), u_2(x), \dots, u_n(x), \dots$ of the function $u(x)$ can be completely determined if we set

$$u_0(x) = f(x)$$
$$u_1(x) = \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_0(x)(dt)^{\alpha}$$
$$u_2(x) = \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_1(x)(dt)^{\alpha} \quad (4)$$
$$u_n(x) = \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_{n-1}(x)(dt)^{\alpha}$$

and so on.
The set above equations can be written in compact recurrence scheme as

$$u_0(x) = f(x), \quad (5)$$

and

$$u_{n+1}(x) = \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)}\int_0^x K(x,t)u_n(x)(dt)^{\alpha} \quad (6)$$

Hence, we give the local fractional series solution

$$u(x) = \sum_{n=0}^{\infty} u_n(x).$$

Definition: Local fractional differential operator

The general local fractional differential equation in a local fractional differential operator form

$$L_x^{(2\alpha)}u(x) + R_x^{\alpha}u(x) = f(x). \quad (1)$$

in equation (1) $L_x^{(2\alpha)}$ is local fractional $2\alpha^{th}$ order differential operator, which by the definition reads

$$L_x^{(2\alpha)}s(x) = \frac{d^{\alpha}}{dx^{\alpha}}\left[\frac{d^{n}}{dx^{\alpha}}s(x)\right], \quad (2)$$

and $R_x^{\alpha}s(x) = \frac{d^{\alpha}s(x)}{dx^{\alpha}}]_{x=x_0}$

$$= \lim_{x\to x_0}\left(\frac{\Delta^{\alpha}(s(x)-s(x_0))}{(x-x_0)^{\alpha}}\right) \quad (3)$$

is local fractional α^{th} order differential operator $(0 < \alpha \le 1)$ and $s(x)$ is local fractional continuous.
From equation (1),

$$L_x^{(2\alpha)}u(x) = f(x) - R_x^{(2\alpha)}u(x).$$

Applying the inverse operator $L_x^{(-2\alpha)}$ to both sides of (1) yields

$$L_x^{(-2\alpha)}L_x^{(2\alpha)}u(x) = -L_x^{(-2\alpha)}R_x^{(\alpha)}u(x) + L_x^{(-2\alpha)}f(x). \quad (4)$$

If the inverse differential operator $L_x^{(-2\alpha)}$ exists, according to the local fractional decomposition method mentioned above, we have

$$\begin{cases} u_{n+1}(x) = -L_x^{(-2\alpha)}R_x^{\alpha}u_n(x) \\ u_n(x) = r(x) \end{cases}, \quad (5)$$

where $r(x) = L_x^{(-2\alpha)}f(x)$
and $L_x^{(-2\alpha)}u(x)$

$$= \left(\frac{1}{\Gamma(1+\alpha)}\right)^2 \int_0^x \int_0^{t_2} u(t_1)(dt_1)^{\alpha}(dt_2)^{\alpha}. \quad (6)$$

Finally, we can find a solution in the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \quad (7)$$

V. AN ILLUSTRATIVE EXAMPLE

Several illustrative examples demonstrating the efficiency of the of the suggested local fractional decomposition method are present next

Example- Solve the local fractional Volterraequation

$$u(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u(t)(dt)^{\alpha}$$

Consider the solution in the series form

$$u(x) = \sum_{i=0}^{\infty} u_n(x)$$

Then substituting this series into the given equation, we have that

$$\sum_{i=0}^{\infty} u(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}\left(\sum_{i=0}^{\infty} u_n(x)\right)(dt)^{\alpha}$$

Now decomposing the different terms in the following manner, we get a set of solutions

$$u_0(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}$$

$$u_1(x) = \frac{1}{\Gamma(1+\alpha)}\int_0^x \frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)}u_0(dt)^{\alpha}$$

$$= \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(x-t)^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} (dt)^\alpha$$
$$= -\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}$$
$$u_2(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(x-t)^\alpha}{\Gamma(1+\alpha)} u_1 (dt)^\alpha$$
$$= \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(x-t)^\alpha}{\Gamma(1+\alpha)} \left(-\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) (dt)^\alpha$$
$$= \frac{t^{5\alpha}}{\Gamma(1+5\alpha)}$$

Continuing in this way we obtain a series

$$u(x) = \sum_{i=0}^{\infty} u_n(x)$$
$$= \sum_{i=0}^{\infty} (-1)^n \frac{x^{(2n+1)\alpha}}{\Gamma(1+(2n+1)\alpha)}$$
$$= \sin_\alpha (x^\alpha)$$

which is the solution of the given local fractional integral equation. This result is similar to the Picard's successive approximation method for the local fractional Volterra integral equation.

Example.- Let us consider the local fractional heat conduction equation with no heat generation in fractal media and dimensionless variables, which reads

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} \quad (1)$$

subject to the following fractal initial boundary conditions

$$u_x^{(\alpha)}(x,0) = 0, u(x,0) = E_\alpha(x^\alpha),$$
$$(0 \le x \le l), \quad (2)$$

where in equation (2),

$u(x,t) = T(x,t)$ is the temperature field.

Hence, the recurrence formula takes the form

$$\begin{cases} u_{n+1}(x,t) = L_t^{(-\alpha)} L_{xx}^{(2\alpha)} u_n(x,t) \\ u_0(x,t) = E_\alpha(x^\alpha) \end{cases} \quad (3)$$

We can develop a solution in a form of local fractional series, namely

$$u(x,t) = \lim_{i \to \infty} \sum_{i=0}^{n} u_i(x,t)$$
$$= E_\alpha(x^\alpha) \lim_{i=0} \left(\sum_{i=0}^{n} \frac{t^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)} \right) \quad (4)$$

$$= E_\alpha(x^\alpha) \sinh_\alpha(t^n)$$

where

$$\sinh_\alpha t^\alpha = \sum_{k=0}^{\infty} t^{\alpha(2k+1)} / \Gamma(1+(2k+1)\alpha)$$

is a hyperbolic cosine function defined on a Cantor set,

and $E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} x^{\alpha k} / \Gamma(1+k\alpha)$ is the Mittag-Leffler function defined on a Cantor set.

VI. CONCLUSION

In this paper, we have solved the local fractional integral equation and local fractional differential equations by using local fractional decomposition method and local fractional differential operators. The local fractional decomposition method fouses especially on the approximation methodology for processing local fractional equations. This method also been explained using two illustrative problems demonstrating its accuracy and reliabilities.

REFERENCES

[1] Majumder, M., et al., Nanoscale Hydrodynamics – Enhanced Flow in Carbon Nanotubes,Nature, 438 (2005), pp. 44, doi: 10.1038/438044a, 7064

[2] Xuan, Y., Roetzel, W., Conceptions for Heat Transfer Correlation of Nanofluids,Int. J. Heat Mass Transfer, 43 (2000), 19, pp. 3701-3707

[3] Adomian, G., A review of the decomposition method and some recent results for nonlinear equation,Mathematical and Computer Modelling, 13 (1990), 7, pp. 17-43

[4] Adomian, G., Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Dordrecht, Netherland, 1994

[5] Wazwaz, A. M., A reliable technique for solving the weakly singular second kind Volterra-type integral equations,Applied mathematics and computation, 80 (1996), pp. 287-299

[6] Nadeem, S., Akbar, N., Effects of heat transfer on the peristaltic transport of MHD Newtonian fluid with variable viscosity: Application of Adomian decomposition method,Communications in Nonlinear Science and Numerical Simulation, 14 (2009), 11, pp. 3844-3855