Integrated Intelligent Research (IIR)

International Journal of Computing Algorithm
Volume: 03 Issue: 02 June 2014 Pages:114-116
ISSN: 2278-2397

On Strong Metric Dimension of Diametrically Vertex Uniform Graphs

Cyriac Grigorious[1],Sudeep Stephen[2], Albert William[3]
Department of Mathematics, Loyola College, Chennai, India

Abstract– A pair of vertices u, v is said to be strongly resolved by a vertex s, if there exist at least one shortest path from s to u passing through v, or a shortest path from s to v passing through u. A set W⊆V, is said to be a strong metric generator if for all pairs u, v ∉ W, there exist some element s ∈ W such that s strongly resolves the pair u, v. The smallest cardinality of a strong metric generator for G is called the strong metric dimension of G. The strong metric dimension (metric dimension) problem is to find a minimum strong metric basis (metric basis) in the graph. In this paper, we solve the strong metric dimension and the metric dimension problems for the circulant graph C (n, $\pm\{1, 2 \ldots j\}$), $1 \le j \le \lfloor n/2 \rfloor$, $n \ge 3$ and for the hypercubes. We give a lower bound for the problem in case of diametrically uniform graphs. The class of diametrically uniform graphs includes vertex transitive graphs and hence Cayley graphs.

Keywords- Strong metric basis; strong metric dimension; circulant graphs; hypercubes; diametrically uniform graphs;

I. INTRODUCTION

A generator of a metric space is a set W of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of W. Given a simple and connected graph $G = (V, E)$, we consider the metric d: $V \times V \to R^+$, where $d(x, y)$ is the length of a shortest path between x and y. (V, d) is clearly a metric space. A vertex $v \in V$ is said to distinguish two vertices x and y if $d(v, x) = d(v, y)$. A set $W \subseteq V$ is said to be a metric generator for G if any pair of vertices of G is distinguished by some element of W. A minimum generator is called a metric basis, and its cardinality the metric dimension of G, denoted by *dim(G)*. For an ordered metric generator $W = \{w_1, w_2 \ldots w_k\}$ of $V(G)$, we refer to the k-vector (ordered k-tuple) $c(v|W) = (d(v, w_1), d(v, w_2) \ldots d(v, w_k))$ as the metric code (or representation) of v with respect to W. The problem of finding the metric dimension of a graph was first studied by Harary and Melter[2]. Slater described the usefulness of these ideas into long range aids to navigation [3]. Melter and Tomescu [5] studied the metric dimension problem for grid graphs. This problem has been studied for trees, multi- dimensional grids [6] and Torus Networks [7]. Also, these concepts have some applications in chemistry for representing chemical compounds [8] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures. Other applications of this concept to navigation of robots in networks and other areas appear in [9, 10]. Khuller *et al.* [6] describe the application of this problem in the field of computer science and robotics. Garey and Johnson [11] proved that this problem is NP-complete for general graphs by a reduction from 3-dimensional matching. Recently Manuel *et al.* [12] have proved that the metric dimension problem is NP-complete for bipartite graphs by a reduction from 3-SAT, thus narrowing down the gap between the polynomial classes and NP-complete classes of the metric dimension problem. Some variations on resolvability or location number have been appearing in the literature, like those about conditional resolvability [15], locating domination [16], resolving domination [17] and resolving partitions [18, 19, 20, 21]. Given the metric basis of some graph and the representation of all the vertices, it is not possible to trace back the graph uniformly. To overcome this, a new variant of metric basis was introduced called strong metric basis.

Two vertices u, v are said to be strongly resolved by the vertex s, if there exist at least one shortest path from s to u passing through v, or a shortest path from s to v passing through u. A subset $W \subseteq V$, is said to be a strong metric generator if for all pairs $u, v \notin W$, there exist some element $s \in W$ such that s strongly resolves the pair u, v. The smallest cardinality of a strong metric basis for G is called the strong metric dimension of G denoted by $sdim(G)$.It is proved that $sdim(G) = n - 1$ if and only if G is the complete graph of order n. For the cycle C_n of order n, $sdim(C_n) = \lceil n/2 \rceil$ and if T is a tree, its strong metric dimension equals the number of leaves of T minus 1 [18]. It is known that the problem of computing the strong metric dimension of a graph is NP-hard [22]. The aim of this paper is to find the metric dimension and strong metric dimension of circulant graphs.

II. DIAMETRICALLY VERTEX UNIFORM GRAPHS

Following symmetrical graphs such as Cayley graphs, vertex transitive graphs, we discuss about the strong metric diemnsion of a new class of sym- metrical graphs called diametrically uniform graphs. The class of diamet- rically uniform graphs includes vertex transitive graphs and hence Cayley graphs. For each vertex u of a graph G, the maximum distance d(u, v) to any other vertex v of G is called its eccentricity and is denoted by ecc(u). In a graph G, the maximum value of eccentricity of vertices of G is called the diameter of G and is denoted by λ. In a graph G, the minimum value of eccentricity of vertices of G is called the radius of G and is denoted by ρ. In a graph G, the set of vertices of G with eccentricity equal to the radius ρ is called the center of G and is denoted by Z(G). Let G be a graph with diameter λ. A vertex v of G is said to be diametrically opposite to a vertex u of G, if d(u, v) =

λ. A graph G is said to be a diametrically vertex uniform (DVU) graph if every vertex of G has at least one diametrically opposite ver- tex. The set of diametrically opposite vertices of u in G is denoted by D(u). An edge (u^*, v^*) of G is said to be diametrically opposite to an edge (u, v) of G, if $d(u, u^*) = \lambda$ and $d(v, v^*) = \lambda$. A graph G is said to be a diametrically edge uniform (DEU) graph if every edge of G has at least one diametrically opposite edge. The set of diametrically opposite edges of (u, v) in G is de- noted by D(u, v). A complete graph is a diametrically vertex uniform graph with the diameter 1.

Lemma 2.1. Let G be a any graph, if two vertices u and v are at diameter distance, then either $u \in W$ or $v \in W$ where W is a strong metric generator.Using the above lemma, we can arrive at a lower bound for the strong metric dimension of all diametrically vertex uniform graphs.

Theorem:Let G be a all diametrically vertex uniform graph of order n. Then sdim $\geq \lceil \frac{n}{2} \rceil$.

Proof. In a all diametrically vertex uniform graph, for all vertices $v \in V(G)$, there exist at least on vertex at diameter distance. Hence by Lemma 2.1 sdim $\geq \lceil \frac{n}{2} \rceil$ for all circulant graphs.

III. CIRCULANT GRAPHS

The circulant graph is a natural generalization of the double loop network and was first considered by Wong and Coppersmith [23]. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [24]. It is also used in VLSI design and distributed computation [25, 26, 27]. The term circulant comes from the nature of its adjacency matrix. A matrix is circulant if all its rows are periodic rotations of the first one. Circulant matrices have been employed for designing binary codes [24]. Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. [21]. Every circulant graph is a vertex transitive graph and a Cayley graph [25]. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [24, 25]. An undirected circulant graph, denoted by $C(n, \pm\{1, 2 \ldots j\})$, $1 \leq j \leq \lfloor n/2 \rfloor$, $n \geq 3$ is a graph with vertex set $V = \{0, 1 \ldots n-1\}$ and edge set $E = \{(i, j): |j-i| \equiv s (mod n), s \in \{1, 2 \ldots j\}\}$. It is clear that $C(n, \pm 1)$ is an undirected cycle C_n and $C(n, \pm\{1, 2 \ldots \lfloor n/2 \rfloor\})$ is the complete graph K_n. C_n is a subgraph of $C(n, \pm\{1, 2 \ldots j\})$ for every j, $1 \leq j \leq \lfloor n/2 \rfloor$, and is sometimes referred to as the outer cycle.

A. A lower bound for strong metric dimension

By Theorem 2.2, we can obtain the following lower bound for a circulant graph

Theorem: *Let* $G = C(n, \pm\{1, 2 \ldots j\})$ be a circulant graph of order n. Then sdim $\geq \lceil \frac{n}{2} \rceil$.

Proof. In a circulant grpahs for all vertices $v \in V(G)$, there exist at least on vertex at diameter distance. Hence by Lemma 2.1 sdim $\geq \lceil \frac{n}{2} \rceil$ for all circulant graphs.

B. An upper bound for strong metric dimension

An upper bound can be obtained by exhibiting a strong resolving set.

Theorem: *Let* $G = C(n, \pm\{1, 2 \ldots j\})$ *be a circulant graph of order n and* $n \equiv i \pmod{2j}$, $2 \leq i \leq 2j + 1$. *Then* $sdim = \lceil n/2 \rceil + \lfloor i/2 \rfloor - 1$. Labels of vertices are subjected to wraparound convention.

Proof. We prove the theorem as follows. First we prove that $s\dim(G) \leq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor - 1$

Let $V = \{1, 2, \ldots n\}$ be the vertex set of G. We prove that the subset $W = \{1, 2, \ldots \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor - 1\}$ of the vertex set V is a strong resolving set of G. Consider two arbitrary vertices *u* and *v*. without loss of generality assume that $u<v$. The vertex $u-dj \in W$ and both *u* and *v* are at different distances. Shortest path from $u-dj \in W$ to *v* passes through u if $|v-u| \leq i$ or shortest path from $u-dj \in W$ to u passes through *v* if $|u-v| < i$ Hence $s\dim(G) \leq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor - 1$.

Since *G* is vertex transitive graph, any set of consecutive vertices with the same cardinality $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor - 1$ is a strong resolving set of G.

We now prove that $s\dim(G) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor - 1$. For a vertex *v* there are exactly $i-1$ vertices at diameter distance. Because of this, it can be easily shown that any subset of vertex set with cardinality $\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{i}{2} \right\rfloor - 2$ does not resolve *G*

IV. HYPER CUBES

An r-dimensional hypercube Q^r has vertices represented by all the binary $r-$tuples with two vertices being adjacent if and only if their corresponding $r-$tuples differ in exactly one position. The decimal representation of the vertices is given by the elements of the set $\{0, 1, 2 \ldots 2^r - 1\}$. For convenience we use the symbol $x + 1$ instead of x, and therefore, the set of labels of the vertices is $\{1, 2, \ldots, 2^r\}$.

Theorem : Let G be an r-dimensional hypercube Q^r, then $sdim(Q^r) = 2^{r-1}$.

Proof: Since an r-dimensional hypercube Q^r is diametrically vertex uniform, from Theorem 2.2 we get $sdim(Q^r) \geq 2^{r-1}$. To prove the upper bound, we show that the subset

International Journal of Computing Algorithm
Volume: 03 Issue: 02 June 2014 Pages:114-116
ISSN: 2278-2397

W $= \{1, 2, ... 2^{r-1}\}$ of vertex set of the r-dimensional hypercube Q^r is a strong resolving metric basis. Consider two arbitrary vertices u, v $\notin$ W. Without loss of generality assume that u < v. The vertex $u' = u - 2^{r-1}$ is a vertex in W and it is at distance 1 from u. Clearly the shortest path from u' to v passes through u. Hence u and v are strongly resolved. This implies that $sdim(Q^r) \leq 2^{r-1}$. Hence we get that $sdim(Q^r) = 2^{r-1}$

REFERENCES

[1]. G Chartrand, L. Eroh, M. Johnson, O. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Appl. Math. 105 (2000), no. 1-3, 99-113.

[2]. arary,F., Melter, R. A.: On the metric dimension of a graph. *Theory of Computing Systems, Ars Com-binatoria*, 2 191-195 (1976)

[3]. Slater, P. J.: Leaves of trees. *Proc. 6th Southeastern Conference on Com- binatorics, Graph Theory, and Computing, Congressus Numerantium*, 14, 549-559 (1975)

[4]. S. Yushmanov, Estimates for the metric dimension of a graph in terms of the diameters and the number of vertices, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 103 (1987) no. 1, 68-70.

[5]. R.A Melter, I. Tomescu, Metric bases in digital geometry, Computer Vision, Graphics, and Image Processing 25 (1984) 113-121.

[6]. S. Khuller,B. Ragavachari, A. Rosenfeld, Landmarks in graphs, Discrete Applied Mathematics 70 (1996) 217-229.

[7]. P. Manuel, R. Bharati, I. Rajasingh, C. Monica, Landmarks in torus networks, Journal of Discrete Mathematical Sciences Cryptography 9 (2) (2006) 263-271.

[8]. M.A. Johnson, Structure-activity maps for visualizing the graph vari- ables arising in drug design, Journal of Biopharmaceutical Statistics 3 (1993) 203-236.

[9]. Caceres, J., Hernando, C., Mora, M., Pelayo I. M., Puertas, M. L., Seara, C.: On the metric dimension of some families of graphs. *Electronic Notes in Discrete Mathematics*, 22, 129-133 (2005)

[10]. Hulme, B. L., Shiver, A. W., Slater, P. J.: A Boolean algebraic analysis of fire protection. *Algebraic and Combinatorial Methods in Operations Research*, 95, 215-227 (1984)

[11]. M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP Completeness, W.H. Freeman and Company, 1979.

[12]. Paul Manuel, Mostafa Abd-El-Barr, Indra Rajasingh and Bharati Ra- jan, An Efficient Representation of Benes Networks and its Applications, Journal of Discrete Algorithm 6 (2008) 11-19

[13]. M. Paul, R. Bharati, R. Indra, M. Chris, On Minimum Metric Dimension of Honeycomb Networks, Journal of Discrete Algorithms, 6 (2008) 20-27.

[14]. M. Paul, R. Bharati, R. Indra, M. M. Chris Monica, Landmarks in binary tree derived architectures, Ars Combinatoria, 99 (2011) 473-486.

[15]. Saenpholphat, V., Zhang, P: Conditional resolvability in graphs: a sur- vey. *International Journal of Mathematics and Mathematical Sciences*, 38, 1997-2017 (2004)

[16]. Haynes, T. W., Henning, M., Howard, J.: Locating and total dominating sets in trees. *Discrete Applied Mathematics*, 154, 1293-1300 (2006)

[17]. Brigham, R. C., Chartrand, G.,Dutton, R. D., Zhang, P.: Resolving domination in graphs. *Mathematica Bohemica*, 128(1), 25-36 (2003)

[18]. Caceres, J., Hernando, C., Mora, M., Pelayo I. M., Puertas, M. L., Seara, C., Wood,D. R.: On the metric dimension of Cartesian product of graphs. *SIAM Journal of Discrete Mathematics*, 21(2), 273-302 (2007)

[19]. Chartrand, G., Salehi, E., Zhang, P.: The partition dimension of a graph. *Aequationes Mathematicae*, 59, 45-54 (2000).

[20]. Fehr, M., Gosselin, S., Oellermann, O. R., The partition dimension of Cayley digraphs. *Aequationes Mathematicae*, 71, 1-18 (2006).

[21]. Yero, G., Rodreguez, J. A.: A note on the partition dimension of Carte- sian product graphs. *Applied Mathematics and Computation*, 217(7), 3571-3574, (2010)

[22]. Andrs Sebo, Eric Tannier, On Metric Generators of Graphs, Mathemat- ics of Operations Researc. 29 (2004) No. 2, 383393.

[23]. Wong, G.K., Coppersmith, D.A.: A combinatorial problem related to multimodule memory organization. *J. Assoc. Comput. Machin.*, 21, 392 - 401, (1994)

[24]. Boesch and Wang,J.: Reliable circulant networks with minimum trans- mission delay. *IEEE Transactions on Circuit and Systems*, 32, 1286 -1291, (1985)

[25]. Bermond, J.C., Comellas, F., Hsu, D.F.: Distributed loop computer networks. *A survey journal of Parallel and Distributed Computing*, 24, 2 - 10, (1995)

[26]. Beivide, R., Herrada, E., Balcazar, J.L., Arruabarrena, A.: Optimal dis- tance networks of low degree for parallel computers. *IEEE Transactions on Computers*, 40, 1109 - 1124, (1991)

[27]. Wilkov, R.S.: Analysis and design of reliable computer networks. *IEEE Transactions on Communications*, 20, 660 - 678, (1972)

[28]. Chris Monica: Minimum Metric Dimension of Certain Interconnection Networks. *Ph.D. Thesis submitted to the University of Madras*, Septem- ber 2007.

[29]. Muhammad Imrana, A.Q. Baig, Syed Ahtsham Ul Haq Bokhary, Imran Javaid : On the metric dimension of circulant graphs. *Applied Mathe- matics Letters*, textbf25, 320-325, (2012).