Skew Chromatic Index of Theta Graphs

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Abstract—A two edge coloring of a graph \( G \) is said be a skew edge coloring if no two edges of \( G \) are assigned the same unordered pair of colors. The least number of colors required for a skew edge coloring of \( G \) is called its skew chromatic index denoted by \( s(G) \). This article provides a method for skew edge coloring of uniform theta and quasi-uniform theta graphs as two component colorings by defining two mappings \( f \) and \( g \) from the set \( E(G) \) to the set of colors \( \{1, 2, 3, \ldots, k\} \). The minimum number of colors \( k \) which is known as the skew chromatic index is determined depending upon the number of edges of \( G \). This work also proves that the bound on the skew chromatic index \( s(G) \) is sharp for the family of graphs considered for skew edge coloring.

Keywords—edge coloring; skew chromatic index; generalized theta graph; uniform theta graph; quasi-uniform theta graph

I. INTRODUCTION

Edge coloring problems in graphs arise in several computer science disciplines [1]. One of which is register allocation during code generation in a computer programming language compiler. This article considers a finite and simple graph \( G \) whose vertex set is \( V \) and edge set is \( E \). A proper edge coloring of \( G \) is an assignment of colors to the edges such that adjacent edges are assigned different colors [2]. The least number of colors required for such an edge coloring of \( G \) is its chromatic index denoted by \( \chi'(G) \).

Vizing [3] has shown that for any simple graph \( G \), \( \chi'(G) \) is either \( \Delta(G) \) or \( \Delta(G) + 1 \), where \( \Delta(G) \) is the highest degree of a graph \( G \). Edge coloring problem has been proved to be NP-complete by Holzer [4]. This work includes skew edge coloring problems which are motivated from the analysis of skew Room squares [5]. Marshall introduced skew chromatic index and the related concepts [6]. The skew chromatic index \( s(G) \) have already been determined for comb, ladder, Mobius ladder and circular ladder graphs. [7, 8]. An assignment of color pairs of the form \( (a_i, b_i) \) to each edge \( e_i \) of \( G \) such that (i) the \( a_i \)’s and the \( b_i \)’s separately form component edge colorings of \( G \) and (ii) all these pairs are distinct is called a skew edge coloring of \( G \). All the notations and definitions which appear in this article are the same as in [9].

II. LOWER BOUND ON SKEW CHROMATIC INDEX

The component coloring of a skew edge coloring is itself an edge coloring and therefore we have \( s(G) \geq \chi'(G) \).

But Vizing’s theorem states that for any simple graph, \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \). Hence \( s(G) \geq \Delta(G) \). Suppose if \( k \) colors are used for skew edge coloring, then there are \( \binom{k + 1}{2} \) unordered pairs of colors and this number must be greater than or equal to the number of edges in \( G \). Let \( k(m) \) denote the smallest integer \( k \) satisfying \( \binom{k + 1}{2} \geq m \) where \( m \) denotes the number of edges in \( G \). Thus the best lower bound for \( s(G) \) is \( s(G) \geq \max \{\Delta(G), k[E(G)]\} \) [6]. This work mainly proves that the bound on the skew chromatic index given here is sharp for uniform theta and quasi-uniform theta graphs.

III. UNIFORM THETA GRAPHS

Definition 3.1: [10] A graph \( \Theta(s_1, s_2, \ldots, s_n) \) which consists of two end vertices namely north pole (N) and south pole (S) joined by \( n \) internally disjoint paths called longitudes (\( L \)) of length greater than one is called a generalized theta graph, and the number of internal vertices in each longitude \( L_i \) being \( s_i, 1 \leq i \leq n \).

Definition 3.2: [10] A uniform theta graph \( \Theta(n, l) \) is a generalized theta graph with \( l \) longitudes \( L_1, L_2, \ldots, L_l \) such that \( |L_1| = |L_2| = \ldots = |L_l| = n \), and \( |L_i| \) being the number of internal vertices in \( L_i \). See Fig. 1.

Fig. 1. (a) \( \Theta(2, 3, 4, 1) \) (b) \( \Theta(n, l) \)

Theorem 3.1: A uniform theta graph \( \Theta(n, l), n \geq 3 \) and \( 3 \leq l \leq 2n - 2 \) is skew edge colorable.
Proof: Consider a uniform theta graph $G = \Theta(n, l)$, $n \geq 3$ and $3 \leq l \leq 2n - 2$. Let $V(G) = \{N | U \{S \cup \{v_i, 1 \leq i \leq nl\} \}$ and $E(G) = \{(Nv_i) \cup \{v_{i'} \cup \{v_{i'}, v_{i''} \cup \{v_{i''}, v_{i'''}, \} \cup \} \}$. Here $V(G) = nl + 2 + 2d[E(G)] = (n + 1)l$.

Define the mappings $f : E(G) \rightarrow \{1, 2, 3, ..., k\}$ and $g : E(G) \rightarrow \{1, 2, 3, ..., k\}$ as follows when $k$ is the smallest positive integer satisfying \[
\frac{(k + 1)}{2} \geq m, \text{ the number of edges of the graph.}
\]

For $1 \leq i \leq l$, define $f(Nv_i) = i$

\[
f(v_{i'), v_{i''}) = \begin{cases} 
2l + i, & 2l + i < k \\
k, & 2l + i = 0 \text{ (mod } k) \\
r, & 2l + i = r \text{ (mod } k) 
\end{cases}
\]

\[
f(v_{i''}, v_{i'''}, v_{i''''}) = \begin{cases} 
2l + i, & 2l + i < k \\
k, & 2l + i = 0 \text{ (mod } k) \\
r, & 2l + i = r \text{ (mod } k) 
\end{cases}
\]

\[
f(v_{i''''}, 2l + i, v_{i''''''}) = \begin{cases} 
2l + i, & 2l + i < k \\
k, & 2l + i = 0 \text{ (mod } k) \\
r, & 2l + i = r \text{ (mod } k) 
\end{cases}
\]

\[
f(v_{i''''''}, S) = \begin{cases} 
2l + i, & 2l + i < k \\
k, & 2l + i = 0 \text{ (mod } k) \\
r, & 2l + i = r \text{ (mod } k) 
\end{cases}
\]

The mapping $f$ assigns the different colors $\{1, 2, 3, ..., l\}$ to the $l$ edges incident to the vertex $N$. The assignment of colors to any two adjacent edges on each of the longitudes $L_i$ and to the edges incident to the vertex $S$ is a proper coloring. Suppose $f(v_{i''''}, v_{i''''''}) = f(v_{i''''', v_{i'''''}})$, then $2l + i$ and $3l + i$ are in the same residue class $[r]$ modulo $k$. This is true if and only if $3l + i = 2l + i \text{ (mod } k)$. This implies $k \mid l$, a contradiction. Thus $f$ defines a proper edge coloring and is called the first component coloring of $\Theta(n, l)$.

For $1 \leq i \leq l$, define $g(Nv_i) = i$

\[
g(v_{i'}, v_{i''}) = \begin{cases} 
1 + i, & 1 + i \leq k \\
r + q, & 1 + i = r \text{ (mod } k) \\
2l + i, & 2l + i < k \\
q - 1, & 2l + i = 0 \text{ (mod } k) \\
r + q, & 2l + i = r \text{ (mod } k) \\
r + q + k, & 2l + i = r \text{ (mod } k) \\
r + q > k 
\end{cases}
\]

\[
g(v_{i''}, v_{i'''}) = \begin{cases} 
1 + i, & 1 + i \leq k \\
r + q, & 1 + i = r \text{ (mod } k) \\
2l + i, & 2l + i < k \\
3l + i, & 3l + i = 0 \text{ (mod } k) \\
q - 1, & 3l + i = r \text{ (mod } k) \\
r + q, & 3l + i = r \text{ (mod } k) \\
r + q + k, & 3l + i = r \text{ (mod } k) \\
r + q > k 
\end{cases}
\]

\[
g(v_{i'''}, v_{i''''}, v_{i''''''}) = \begin{cases} 
1 + i, & 1 + i \leq k \\
r + q, & 1 + i = r \text{ (mod } k) \\
2l + i, & 2l + i < k \\
3l + i, & 3l + i = 0 \text{ (mod } k) \\
q - 1, & 3l + i = r \text{ (mod } k) \\
r + q, & 3l + i = r \text{ (mod } k) \\
r + q + k, & 3l + i = r \text{ (mod } k) \\
r + q > k 
\end{cases}
\]

\[
g(v_{i''''''}, S) = \begin{cases} 
1 + i, & 1 + i \leq k \\
r + q, & 1 + i = r \text{ (mod } k) \\
2l + i, & 2l + i < k \\
3l + i, & 3l + i = 0 \text{ (mod } k) \\
q - 1, & 3l + i = r \text{ (mod } k) \\
r + q, & 3l + i = r \text{ (mod } k) \\
r + q + k, & 3l + i = r \text{ (mod } k) \\
r + q > k 
\end{cases}
\]

where $q$ is the quotient in each case mod $k$. Thus $g$ defines a proper edge coloring and is called the second component coloring of $\Theta(n, l)$. The two component colorings of $\Theta(n, l)$ defined by $f(E(G))$ and $g(E(G))$ is as follows. The first $k'$ edges are assigned the colors of the form $(c, c) = 1, 2, 3, ..., k$. In the second set of $k'$ edges, the first $k - 1$ edges are assigned the colors of the form $(c, c + 1)$, $c = 1, 2, 3, ..., k - 1$. The $k^{th}$ edge is assigned the color $(k, 1)$ as the ordered pair $(k, k + 1)$ is not permissible. In the third set of $k'$ edges, the first $k - 2$ edges are assigned the colors of the form $(c, c + 2)$, $c = 1, 2, 3, ..., k - 2$. The $(k - 1)^{th}$ and the $k^{th}$ edge are assigned the colors $(k - 1, 1)$ and $(k, 2)$ respectively as the ordered pairs $(k - 1, k + 1)$ and $(k, k + 2)$ are not permissible. Thus the pairs of colors $(f, g)$ assigned to each edge in $\Theta(n, l)$ are all distinct and forms the skew edge coloring of $G$. See Fig. 2.

**Theorem 3.2**: Let $G = \Theta(n, l)$, $n \geq 3$ and $3 \leq l \leq 2n - 2$ be a uniform theta graph. Then $s(G) = \left\lceil \frac{1 + \sqrt{4(n + 1)}}{2} \right\rceil$.
Proof: Since there are \((n+1)!\) edges in \(G\), its skew edge coloring will require at least \((n+1)!\) pairs of colors. If \(k'\) colors are used for coloring, then there are \(\binom{k+1}{2}\) unordered pairs. This \(\binom{k+1}{2} \geq |E(G)|\). Therefore fix \(k'\) in such a way that \(\binom{k+1}{2} \geq (n+1)!\). It follows that \(k' + k - (2n + 2t) \geq 0\). This is a quadratic equation in \('k'\). Its solution gives the required value of \(s(G) = k = \left\lfloor \frac{-1 + \sqrt{1 + 8(n + 1)!}}{2} \right\rfloor\).

By comparing with the bound on \(s(G)\) as discussed in section II, the skew chromatic index obtained is equivalent to \(k(|E(G)|)\) which is an optimal solution for the skew chromatic index of uniform theta graphs \(\Theta(n, l)\) for \(n \geq 3\) and \(3 \leq l \leq 2n - 2\). Hence for a given \(\Theta(n, l)\), we immediately obtain its \(s(G)\) by a simple manipulation.

IV. QUASI UNIFORM THETA GRAPHS

In this section, we determine the skew chromatic index of quasi-uniform theta graphs for \(n \geq 3\) and \(3 \leq l \leq 2n - 2\).

Definition 4.1: [10] A quasi-uniform theta graph is a generalized theta graph with \(l\) longitudes \(L_1, L_2, \ldots, L_l\) such that \(|L_1| = |L_2| = \ldots = |L_m| = l\), and \(|L_i|\) being the number of internal vertices in \(L_i\). It is said to be even or odd according as \([L_1, \ldots, L_l]\) is even or odd. It is obvious that every uniform theta graph is also an even quasi-uniform theta graph. See Fig. 3.

Theorem 4.1: Let \(G\) be a quasi-uniform theta graph (even or odd) with \(l\) longitudes \(L_1, L_2, \ldots, L_l\) and \(n\) vertices on each \(L_i, 1 \leq i \leq l - 1\) and \(n + t\) vertices on \(L_l\), where \(t = |L_l| - k\), \(n \geq 3\) and \(3 \leq l \leq 2n - 2\). Then \(G\) admits skew edge coloring and its skew chromatic index is \(s(G) = \left\lfloor \frac{-1 + \sqrt{1 + 8(n + 1)! + 2l}}{2} \right\rfloor\).

Proof: Let \(G = \Theta(s_1, s_2, \ldots, s_l, t_1)\) where \(s_1 = s_2 = \ldots = s_m = n\), \(s_j = n + t\) be a quasi-uniform theta graph with \(n \geq 3\) and \(3 \leq l \leq 2n - 2\).

Let \(V(G) = \{N\} \cup \{S\} \cup \{v_i, 1 \leq i \leq n\} \cup \{v_{i,j}, 1 \leq i \leq l\}\) and \(E(G) = \{ (v_{ij}, v_{i,j+1}) \cup \{v_{i,j}, v_{i+1,j}\} \cup \ldots \cup \{v_{i,j}, v_{i,n,t-1}\}, \quad 1 \leq i \leq l\} \cup \{v_{i,j}, v_{i,j+1}, 1 \leq i \leq l - 1\} \cup \{v_{i,j}, v_{i,n,t}, 0 \leq i \leq l - 1\} \cup \{v_{i,n,t}, S\} \).

Here \(|V(G)| = nl + t + 2\) and \(|E(G)| = (n + 1)!l + t\).

Define the mappings \(f : E(G) \to \{1, 2, 3, \ldots, k\}\) and \(g : E(G) \to \{1, 2, 3, \ldots, k\}\) as follows where \(k\) is the smallest positive integer satisfying \(\binom{k+1}{2} \geq (n + 1)! + t\).

For \(1 \leq i \leq l\), define
\[
\begin{align*}
f(Nv_i) &= i, \\
f(v_{i,j}, v_{i,j+1}) &= \left\lfloor \frac{2l + i}{k} \right\rfloor, \\
f(v_{i,j}, v_{i+1,j}) &= \left\lfloor \frac{2l + i}{k} \right\rfloor, \\
f(v_{i,j}, v_i) &= \left\lfloor \frac{2l + i}{k} \right\rfloor, \\
f(v_{i,j}, v_{i+1,j+1}) &= \left\lfloor \frac{2l + i}{k} \right\rfloor, \\
f(v_{i,j}, v_{i+1,j+1}) &= \left\lfloor \frac{2l + i}{k} \right\rfloor.
\end{align*}
\]

The following two cases arise when we color the remaining \(\frac{t - 1}{2}\) edges:
\[
\begin{align*}
& f(v_{i,j}, v_{i+1,j+1}) = \left\lfloor \frac{2l + i}{k} \right\rfloor, \\
& f(v_{i,j}, v_{i+1,j+1}) = \left\lfloor \frac{2l + i}{k} \right\rfloor, \\
& f(v_{i,j}, v_{i+1,j+1}) = \left\lfloor \frac{2l + i}{k} \right\rfloor.
\end{align*}
\]

Thus "f" defines a proper edge coloring and is called the first component coloring of the quasi-uniform theta graph.
\[
\begin{align*}
g(Nv_i) &= i \\
g(v_{1i}, v_{1i}) &= \begin{cases} 
1 + i, & l + i \leq k \\
r + q, & l + i \equiv r \pmod{k} \\
2l + i, & 2l + i \leq k 
\end{cases} \\
g(v_{2i}, v_{2i}) &= \begin{cases} 
q - 1, & 3l + i \equiv 0 \pmod{k} \\
r + q, & 3l + i \equiv r \pmod{k}, r + q \leq k \\
r + q - k, & 3l + i \equiv r \pmod{k}, r + q > k 
\end{cases} \\
\vdots \\
g(v_{(c-2)i}, v_{(c-1)i}) &= \begin{cases} 
q - 1, & (n-1)l + i \equiv 0 \pmod{k} \\
r + q, & (n-1)l + i \equiv r \pmod{k}, r + q \leq k \\
r + q - k, & (n-1)l + i \equiv r \pmod{k}, r + q > k 
\end{cases}
\end{align*}
\]

For \(1 \leq i \leq l - 1\),
\[
\begin{align*}
g(v_{(c-2)i}, S) &= \begin{cases} 
q - 1, & n + l + i \equiv 0 \pmod{k} \\
r + q, & n + l + i \equiv r \pmod{k}, r + q \leq k \\
r + q - k, & n + l + i \equiv r \pmod{k}, r + q > k
\end{cases}
\end{align*}
\]

For \(0 \leq i \leq \left\lfloor \frac{t}{2} \right\rfloor\),
\[
\begin{align*}
g(v_{ct}, v_{ct+i}) &= \begin{cases} 
q - 1, & n + l + 2i \equiv 0 \pmod{k} \\
r + q, & n + l + 2i \equiv r \pmod{k}, r + q \leq k \\
r + q - k, & n + l + 2i \equiv r \pmod{k}, r + q > k
\end{cases}
\end{align*}
\]

The following two cases arise when we color the remaining \(\left\lceil \frac{t-1}{2} \right\rceil\) edges \(\{v_{ct}, v_{ct+1}, v_{ct+2}, \ldots, v_{ct+i}, \ldots, v_{ct+2}\} \) on the longitude \(L_t\).

Case (i): \(r + q, n + l + 1 - (2i - 1) \equiv 0 \pmod{k}\)
\[
g(v_{ct}, v_{ct+i}) = \begin{cases} 
q - 1, & n + l + 1 \equiv r \pmod{k}, r + q \leq k \\
r + q, & n + l + 1 \equiv r \pmod{k}, r + q > k
\end{cases}
\]

Case (ii): \(r + q, n + l + i - (2i - 1) \equiv 0 \pmod{k}\)
\[
g(v_{ct}, v_{ct+i}) = \begin{cases} 
q - 1, & n + l + 1 \equiv r \pmod{k}, r + q \leq k \\
r + q, & n + l + 1 \equiv r \pmod{k}, r + q > k
\end{cases}
\]

where ‘q’ is the quotient in each case \(mod k\). Thus ‘g’ defines a proper edge coloring and is called the second component coloring of the quasi-uniform theta graph. The two component colorings defined by \(f(E(G))\) and \(g(E(G))\) for each edge in the quasi-uniform theta graph \(\Theta (x_1, x_2, \ldots, x_i, \ldots, x_j)\) are all distinct and thus give the skew edge coloring of \(G\). The first ‘k’ edges are assigned the colors of the form \((c, c)\), \(c = 1, 2, 3, \ldots, k\). In the second set of ‘k’ edges, the first \(k - 1\) edges are assigned the colors of the form \((c, c+1)\), \(c = 1, 2, 3, \ldots, k - 1\). The \(k^{th}\) edge is assigned the color \((k, 1)\) as the ordered pair \((k, k + 1)\) is not permissible. In the third set of ‘k’ edges, the first \(k - 2\) edges are assigned the colors of the form \((c, c+2)\), \(c = 1, 2, 3, \ldots, k - 2\). The \((k - 1)^{th}\) and the \(k^{th}\) edge are assigned the colors \((k - 1, 1)\) and \((k, 2)\) respectively as the ordered pairs \((k - 1, k + 1)\) and \((k, k + 2)\) are not permissible. Thus the pairs of colors \((f, g)\) assigned to each edge in the quasi-uniform theta graph are all distinct and forms the skew edge coloring of \(G\). See Fig. 4.

Proceeding as in Theorem 3.2, an optimal \(s(G)\) for quasi-uniform theta graph is obtained as
\[
x(G) = \left\lfloor \frac{-1 + \sqrt{1 + 8m}}{2} \right\rfloor = \left\lfloor \frac{-1 + \sqrt{1 + 8(n + l)f + 2l}}{2} \right\rfloor.
\]

By comparing with the bound on \(s(G)\) as discussed in Section II, the skew chromatic index obtained is equivalent to \(k[E(G)]\) which is an optimal solution for the skew chromatic index of quasi-uniform theta graphs.
V. CONCLUSION

The skew edge coloring and hence the skew chromatic index of uniform theta and quasi-uniform theta graphs is obtained. Determining the skew chromatic index of other interconnection networks is quite interesting. Finding the skew chromatic index of cyclic snakes is under consideration.

References