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σ – Algebra on Cartesian Product of Vertex Measurable Graphs

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Abstract — Let G_1 and G_2 be two simple graphs. Let $(G_1, \Im_1)$ and $(G_2, \Im_2)$ be two vertex measure spaces. In this paper we introduce a σ algebra $\Im_1 \times \Im_2$, which consists of all vertex induced sub graphs of $G_1 \times G_2$, and it contains every vertex measurable rectangle graph of the form $H_1 \times H_2$, $H_1 \in \Im_1$ and $H_2 \in \Im_2$. Here, we prove $\Im_1 \times \Im_2$ is the smallest σ algebra of $G_1 \times G_2$ such that the maps $L_{G_1}: G_1 \times G_2 \to G_1$ and $L_{G_2}: G_1 \times G_2 \to G_2$ defined by $L_{G_1}(H \times K) = H$ and $L_{G_2}(H \times K) = K$ for all vertex measurable graphs H in G_1 and K in G_2 respectively are measurable.

Keywords — vertex measurable graph, vertex measurable rectangle graph

I. Introduction

The authors [3] introduced the concept of vertex measurable graph and proved some results related to this concept. The authors [4] introduced a new operation '$\veebar$' as in definition 2.4 and vertex measurable rectangle graph. The concept of Cartesian product of two measurable spaces was introduced in the field of measure theory. In [2] it has been proved that $\Im_1 \times \Im_2$ is the smallest σ – algebra of subsets of $X \times Y$ such that the maps $P_x: X \times Y \longrightarrow X$ and $P_y: X \times Y \longrightarrow Y$ defined by $P_x(x, y) = x$ and $P_y(x, y) = y$ respectively are measurable. In this paper we develop the graph analog of these concepts.

II. Preliminaries

Definition 2.1
A graph G with p vertices and q edges is called a (p, q) graph, where p and q are respectively known as the order and size of the graph G.

A (p, q) graph with $p = q = 0$ is called an empty graph and is denoted by Φ.

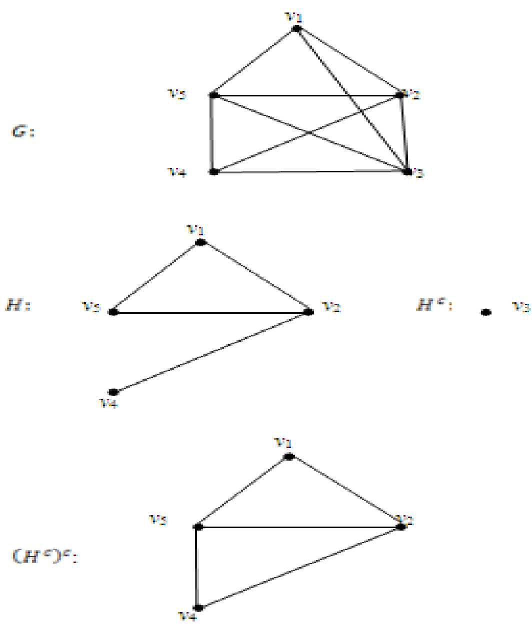
Definition 2.2
Let $G = (V_G, E_G)$ be a graph and $H = (V_H, E_H)$ be a sub graph of G. The vertex complement of H in G is denoted by $H^c_{V_G}$ and it is defined as the sub graph obtained from G by deleting all the vertices of H. Hereafter, we shall use H^c instead of $H^c_{V_G}$. From the examples below it is evident that $H \cup H^c$ is not equal to G. So there is a hidden sub graph of G related to H to overcome this difficulty, a new union was defined in [3].

Definition 2.3
Let $G = (V_G, E_G)$ be a graph and for $S_1 \subset V_G$ and $S_2 \subset V_G$. Let $H_1 = \langle S_1 \rangle$ and $H_2 = \langle S_2 \rangle$ be two vertex induced sub graphs of G. The vertex induced union of H_1 and H_2 is defined as the vertex induced sub graph $\langle S_1 \cup S_2 \rangle$ and is denoted by $H_1 \uplus H_2$.

Definition 2.4
If $H_1 = \langle S_1 \rangle$ and $H_2 = \langle S_2 \rangle$ are two sub graphs of G then $H_1 \veebar H_2 = \langle S_1 - S_2 \rangle$.

Definition 2.5
Let G be a simple graph and let $\Im$ be a collection of vertex induced sub graphs of G together with empty graph Φ is a field if and only if

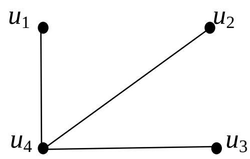
(i) $G \in \Im$
(ii) $H^c \in \Im$ for each $H \in \Im$
(iii) $H_1, H_2, \ldots \in \Im$, then $\uplus H_i \in \Im$ and $\cap H_i \in \Im$



Definition 2.6
Let $G_1 = (V_{G_1}, E_{G_1})$ and $G_2 = (V_{G_2}, E_{G_2})$ be two simple graphs. The Cartesian product of G_1 and G_2 denoted as $G_1 \times G_2$ is a graph with vertex set $V_{G_1} \times V_{G_2}$ and two vertices $u = (u_1, v_1)$, $v = (u_2, v_2)$ are said to be adjacent if $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or $v_1 = v_2$ and u_1 is adjacent to u_2 in G_1. That is, $G_1 \times G_2 = (V_{G_1} \times V_{G_2}, E_{G_1 \times G_2})$ where $E_{G_1 \times G_2} = \{\, uv \,/\, u_1 = u_2 \text{ and } v_1 v_2 \in E_{G_2} \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E_{G_1} \}$

Example 2.7

G_1:



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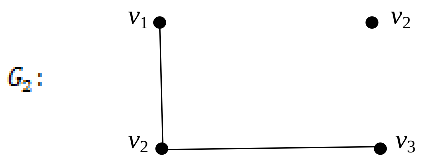

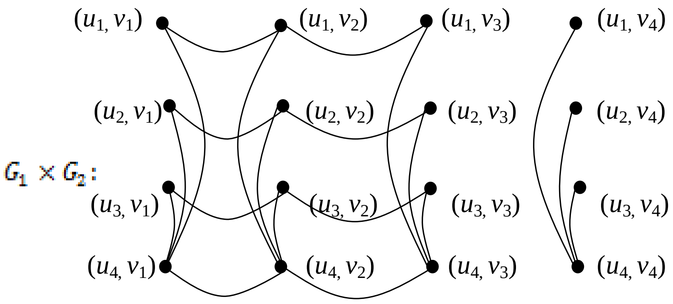

Definition 2.8
Let G be a simple graph. Let $\Im$ be a $\sigma -$ algebra of vertex induced sub graphs of G, then $(G, \Im)$ is called a vertex measure space.

III. VERTEX MEASURABLE RECTANGLE GRAPH

Definition 3.1
Let G_1 and G_2 be any two simple graphs. Let $(G_1, \Im_1)$ and $(G_2, \Im_2)$ be two vertex measurable spaces. Any graph of the form $H_1 \times H_2$, where $H_1 \in \Im_1$ and $H_2 \in \Im_2$, is called a vertex measurable rectangle graph.

Definition 3.2
Let G_1 and G_2 be any two simple graphs. Let N be any vertex induced sub graph of $G_1 \times G_2$. For each vertex u in G_1, define $N_{G_1}^u = \langle V_{G_1}^u \rangle$, where $V_{G_1}^u = \{v_i / v_i \in V(G_2) \text{ and } (u, v_i) \in V(N)\}$. $\langle V_{G_1}^u \rangle$ is the vertex induced sub graph from the vertex set $V_{G_1}^u$. Clearly $N_{G_1}^u$ is a sub graph of G_2. Simply it is denoted as N^u.

Similarly for each vertex v in G_2 define $V_{G_2}^v = \langle V_{G_2}^v \rangle$, where $V_{G_2}^v = \{u_i / u_i \in V(G_1) \text{ and } (u_i, v) \in V(N)\}$. $\langle V_{G_2}^v \rangle$ is the vertex induced sub graph from the vertex set $V_{G_2}^v$. Clearly $N_{G_2}^v$ is a sub graph of G_1.

Lemma 3.3
Let $(G_1, \Im_1)$ and $(G_2, \Im_2)$ be two vertex measure spaces. If $N = H \times K$ is a vertex measurable rectangle graph in $\Im_1 \times \Im_2$, then $N^u = \begin{cases} K \text{ if } u \in V(H) \\ \Phi \text{ if } u \notin V(H) \end{cases}$

Proof:

Let $u \in V(H)$
Then $N^u = \langle V^u \rangle$ where

$$\begin{aligned} V^u &= \{v_i / v_i \in V(G_2) \text{ and } (u, v_i) \in V(N)\} \\ &= \{v_i / v_i \in V(G_2) \text{ and } (u, v_i) \in V(H \times K)\} \\ &= \{v_i / v_i \in V(G_2) \text{ and } u \in V(H) \text{ and } \\ &\quad (u, v_i) \in V(H \times K)\} \\ &= \{v_i / v_i \in V(K)\} \\ &= V(K) \end{aligned}$$

$$N^u = \langle V(K) \rangle = K$$

Hence $N^u = K$ if $u \in V(H)$.
If u is not in $V(H)$, then
$N^u = \langle V^u \rangle$ where

$$\begin{aligned} V^u &= \{v_i / v_i \in V(G_2) \text{ suchthat } (u, v_i) \in V(N)\} \\ &= \{v_i / v_i \in V(G_2) \text{ and } u \notin V(H) \\ &\quad \text{and } (u, v_i) \in V(N)\} \\ &= \{v_i / u \notin V(H) \text{ and } (u, v_i) \in V(H \times K)\} \\ &= \Phi \end{aligned}$$

Therefore, $N^u = \Phi$, if $u \notin V(H)$.
Hence, $N^u = \begin{cases} K \text{ if } u \in V(H) \\ \Phi \text{ if } u \notin V(H) \end{cases}$
Similarly we can prove, for $v \in V(K)$,
$N^v = \begin{cases} H \text{ if } v \in V(K) \\ \Phi \text{ if } v \notin V(K) \end{cases}$

Lemma 3.4
If $H_1 \times K_1$ and $H_2 \times K_2$ are two vertex measurable rectangle graphs in $\Im_1 \times \Im_2$, then
$(H_1 \times K_1) \cap (H_2 \times K_2) = (H_1 \cap H_2) \times (K_1 \cap K_2)$

Proof:
For the proof, we claim that
$V((H_1 \times K_1) \cap (H_2 \times K_2)) = V((H_1 \cap H_2) \times (K_1 \cap K_2))$ and
$E((H_1 \times K_1) \cap (H_2 \times K_2)) = E((H_1 \cap H_2) \times (K_1 \cap K_2))$.
Now, $V((H_1 \times K_1) \cap ((H_2 \times K_2))$

$$\begin{aligned} &= (V(H_1 \times K_1)) \cap V((H_2 \times K_2)) \\ &= (V(H_1) \times V(K_1)) \cap (V(H_2) \times V(K_2)) \\ &= (V(H_1) \cap V(H_2)) \times (V(K_1) \cap V(K_2)) \\ &= V(H_1 \cap H_2) \times V(K_1 \cap K_2) \end{aligned}$$

therefore,
$V((H_1 \times K_1) \cap ((H_2 \times K_2)) = V(H_1 \cap H_2) \times V(K_1 \cap K_2)$
Let $e = uv \in E((H_1 \times K_1) \cap ((H_2 \times K_2))$ where $u = (u_1, v_1)$ and $v = (u_2, v_2)$

$$\Leftrightarrow (u_1, v_1)(u_2, v_2) \in E(H_1 \times K_1) \cap E(H_2 \times K_2)$$

$$\Leftrightarrow (u_1, v_1)(u_2, v_2) \in E(H_1 \times K_1) \text{ and } (u_1, v_1)(u_2, v_2) \in E(H_2 \times K_2)$$

By the definition of Cartesian product,

$[u_1 = u_2 \text{ and } v_1 v_2 \in E(K_1) \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(H_1)]$
and
$[u_1 = u_2 \text{ and } v_1 v_2 \in E(K_2) \text{ or } v_1 = v_2 \text{ and } u_1 u_2 \in E(H_2)]$

that implies and implied by

$[u_1 = u_2 \text{ and } v_1 v_2 \in E(K_1) \text{ and } u_1 = u_2 \text{ and } v_1 v_2 \in E(K_2)]$
or
$[u_1 = u_2 \text{ and } v_1 v_2 \in E(K_1) \text{ and } v_1 = v_2 \text{ and } u_1 u_2 \in E(H_2)]$
or
$[v_1 = v_2 \text{ and } u_1 u_2 \in E(H_1) \text{ and } u_1 = u_2 \text{ and } v_1 v_2 \in E(K_2)]$
or
$[v_1 = v_2 \text{ and } u_1 u_2 \in E(H_1) \text{ and } v_1 = v_2 \text{ and } u_1 u_2 \in E(H_2)]$

that implies and implied by

$[u_1 = u_2 \text{ and } v_1 v_2 \in E(K_1) \cap E(K_2)]$
or

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$[v_1 = v_2$ and $u_1u_2 \in E(H_1) \cap E(H_2)]$
that implies and implied by

$[u_1 = u_2$ and $v_1v_2 \in E(K_1 \cap K_2)]$
or
$[v_1 = v_2$ and $u_1u_2 \in E(H_1 \cap H_2)]$
Hence,
$E((H_1 \times K_1) \cap (H_2 \times K_2)) = E((H_1 \cap H_2) \times (K_1 \cap K_2))$

Theorem 3.5
Let $(G_1, \mathfrak{I}_1)$ and $(G_2, \mathfrak{I}_2)$ be two vertex measure spaces. Let $L_{G_1} : G_1 \times G_2 \to G_1$ and $L_{G_2} : G_1 \times G_2 \to G_2$ be defined by $L_{G_1}(H \times K) = H$ and $L_{G_2}(H \times K) = K$ for all vertex measurable graphs H in G_1 and K in G_2. Then

(i) The maps L_{G_1} and L_{G_2} are measurable.
[for all $H \in \mathfrak{I}_1$ and $K \in \mathfrak{I}_2$, $L_{G_1}{}^{-1}(H) \in \mathfrak{I}_1 \times \mathfrak{I}_2$ and $L_{G_2}{}^{-1}(K) \in \mathfrak{I}_1 \times \mathfrak{I}_2$]
(ii) The σ – algebra $\mathfrak{I}_1 \times \mathfrak{I}_2$ is the smallest σ – algebra of $G_1 \times G_2$, such that (i) holds.

Proof:
Let G_1 and G_2 be two simple graphs. Let $(G_1, \mathfrak{I}_1)$ and $(G_2, \mathfrak{I}_2)$ be two vertex measure spaces. Let $H \in \mathfrak{I}_1$ and $K \in \mathfrak{I}_2$. By lemma 3.3, for any vertex measurable graph $H \times K$ in $\mathfrak{I}_1 \times \mathfrak{I}_2$, $(H \times K)^u = \begin{cases} K \text{ if } u \in V(H) \\ \phi \text{ if } u \notin V(H) \end{cases}$ for all $u \in V(H)$. This follows that $L_{G_1}{}^{-1}(H) \in \mathfrak{I}_1 \times \mathfrak{I}_2$ and $L_{G_2}{}^{-1}(K) \in \mathfrak{I}_1 \times \mathfrak{I}_2$.

Let Ω be any σ – algebra of vertex induced sub graphs of $G_1 \times G_2$ such that L_{G_1} and L_{G_2} are vertex measurable. We have to show that $\Omega \subseteq \mathfrak{I}_1 \times \mathfrak{I}_2$. Let $H \in \mathfrak{I}_1$ and $K \in \mathfrak{I}_2$. By the definition of L_{G_1}, $L_{G_1}(H \times G_2) = H$ which is $H \times G_2 = L_{G_1}{}^{-1}(H)$. Since, $H \times G_2 \in \Omega$, $L_{G_1}{}^{-1}(H) \in \Omega$. Similarly $L_{G_2}{}^{-1}(K) \in \Omega$. By lemma 3.4 $H \times K = (H \cap G_1) \times (K \cap G_2) = (H \times G_2) \cap (G_1 \times K)$. It follows that $\mathfrak{I}_1 \times \mathfrak{I}_2 \subseteq \Omega$. Hence $\mathfrak{I}_1 \times \mathfrak{I}_2$ is the smallest σ – algebra of vertex measurable graphs of $G_1 \times G_2$.

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