Shuffle Exchange Networks and Achromatic Labeling

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Abstract - Design of interconnection networks is an important integral part of the parallel processing or distributed systems. There are a large number of topological choices for interconnection networks. Among several choices, the Shuffle Exchange Network is one of the most popular versatile and efficient topological structures of interconnection networks. In this paper, we have given a new method of drawing shuffle exchange network for any dimension. This has enabled us to investigate some of the topological properties of shuffle-exchange network. Also we give an approximation algorithm for achromatic number of shuffle-exchange network.

Keywords- Interconnection networks, Shuffle Exchange Network, Achromatic Labeling.

I. INTRODUCTION

An interconnection network consists of a set of processors, each with a local memory, and a set of bidirectional links that serve for the exchange of data between processors. A convenient representation of an interconnection network is by an undirected (in some cases directed) graph \( G = (V, E) \) where each processor is a vertex in \( V \) and two vertices are connected by an edge if and only if there is a direct (bidirectional for undirected and unidirectional for directed graphs) communication link between processors. We will use the term interconnection network and graph interchangeably.

A. The Shuffle Exchange Network

Definition - Let \( Q_n \) denote an \( n \)-dimensional hypercube. The \( n \) dimensional shuffle - exchange network, denoted by \( SE(n) \), has vertex set \( V = V(Q_n) \), and two vertices \( x = x_1x_2...x_n \) and \( y = y_1y_2...y_n \) are adjacent if and only if either

(i) \( x \) and \( y \) differ in precisely the \( n^{th} \) bit,
(ii) \( x \) is a left or right cyclic shift of \( y \).

The edge defined by the condition (i) is called an exchange edge, and (ii) is called a shuffle edge. The condition (ii) means that either \( y_1y_2...y_n = x_2x_3...x_nx_1 \) or \( y_1y_2...y_n = x_nx_1x_2...x_{n-2}x_{n-1} \).

As an example, the 8-node shuffle exchange graph is given in Figure 1. The shuffle edges are drawn with solid lines while the exchange edges are drawn with dashed lines [6]. For higher dimension, it is generally understood that drawing shuffle-exchange network is quite challenging.

We have redrawn the Shuffle-Exchange network considering the power set of \( X = \{2^0, 2^1, ..., 2^n - 1\} \) as vertices. This has enabled us to present a good drawing of the Shuffle-Exchange network for any dimension.

II. NEW REPRESENTATION OF THE SHUFFLE EXCHANGE NETWORK

Definition-Let \( X = \{2^0, 2^1, ..., 2^n - 1\} \) and let \( P(X) \) denote the power set of \( X \). We construct a graph \( G^* \) with vertex set \( P(X) \) where

(i) Two nodes \( S \) and \( S' \) are adjacent if and only if \( S \bigtriangleup S' = \{2^0\} \)
(ii) If \( |S| = |S'| = k \), where \( S = \{2^0, 2^1, ..., 2^k\} \) and \( S' = \{2^0, 2^1, ..., 2^k\} \), then \( S \) and \( S' \) are adjacent if and only if \( y_i = (x_i + 1) \mod n \) for all \( 1 \leq i \leq k \).

Figure 1 The 8 - node Shuffle Exchange Network

Figure 2 New Drawing of the Shuffle-Exchange Network

Theorem 1 The two definitions of Shuffle-Exchange network of dimension \( n \) are equivalent. In other words, the graph \( G^* \) constructed in definition 2 is isomorphic to the graph \( G \) given in definition 1.

Proof. Let \( G \) and \( G^* \) be the graphs defined by definition 1 and definition 2 respectively. First we associate with each binary string of length \( n \), a set \( S \in P(X) \).

Let \( u = (u_1, u_2, ..., u_n) \), \( u_i \in \{0, 1\} \) be an arbitrary string. If \( u_i = 1 \), then let \( 2^n - i \in S \). If \( u_i = 0 \), then \( 2^n - i \notin S \). If \( u_i = 0 \) for every \( i \), then \( S = \emptyset \). For example, the string 000 is associated with \( \emptyset \), 001 is associated with \( \{2^n\} \), 010 with \( \{2^n\} \), 011 with \( \{2^0, 2^1\} \), 111 with \( \{2^0, 2^1, 2^2\} \), 110 with \( \{2^0, 2^1\} \), and 100 with \( \{2^0, 2^1\} \).

Define \( f : V(G) \rightarrow V(G^*) \) by \( f(u) = f(u_1, u_2, ..., u_n) = S \) where \( 2^n - i \in S \) if \( u_i = 1 \), \( 2^n - i \notin S \) if \( u_i = 0 \), \( 1 \leq i \leq n \).

Clearly \( f \) is well-defined, for, \( u = v \Rightarrow u_i = v_i \), \( \forall \ i = 1, 2, ..., n \)
\[ f(u) = f(v) \]
\[ S = S' \]

\( f \) is one - one: Let \( S, S' \in V(G^*) \) such that \( S = S' = \{z^1, z^2, \ldots , z^k\} \)
\[ f(u) = f(u') \]
\[ u_i = 1 = u'_i, 1 \leq i \leq k \]
\[ u = u' \]

\( f \) is onto: For every \( S' \in V(G^*) \), there exists \( u = (u_1, u_2, \ldots , u_n) \in V(G) \) such that \( u_i = 1 \) if \( 2^{i-1} \not\in S \) and \( u_i = 0 \) if \( 2^{i-1} \in S, 1 \leq i \leq n \) satisfying \( f(u) = S \).

\( f \) preserves adjacency: Let \( e = uv \) be an exchange edge in \( G \). To prove, \( f(e) = SS' \) is an exchange edge in \( G^* \).

By definition, \( u \) and \( v \) differ in exactly the \( n^{th} \) bit. That is, if \( u_n = 0 \), then \( v_n = 1 \) and vice versa \( \Rightarrow 2^0 \in S \) and \( 2^0 \not\in S' \), that is, if \( S = \{2^0, 2^1, 2^2, \ldots , 2^{n-1}\} \), then \( S' = \{2^1, 2^2, \ldots , 2^{n-1}\} \).

Hence \( S \Delta S' = \{2^0\} \Rightarrow SS' \) is an exchange edge in \( G^* \).

Conversely, let \( SS' \) be an exchange edge in \( G^* \). In other words, if \( S = \{2^0, 2^1, 2^2, \ldots , 2^{n-1}\} \), then \( S' = \{2^1, 2^2, \ldots , 2^{n-1}\} \).

This implies \( u_{n-k} = u_{n-k-1} = \ldots = u_{n-3} = u_{n-2} = 1 = v_{n-1} = v_{n-2} = \ldots = v_{n-k} \), the rest of \( u \) and \( v \) are zero with \( u_n = 1 \) and \( v_n = 0 \).

Hence \( u \) and \( v \) differ in exactly the \( n^{th} \) bit.
\[ uv \] is an exchange edge in \( G \).

Let \( e = uv \) be a shuffle edge in \( G \). Then \( u = (u_0, u_1, \ldots , u_{n-1}) \) is a left (or a right) cyclic shift of \( v = (v_0, v_1, \ldots , v_{n-1}) \), that is, \( v_0v_1 \ldots v_{n-1} = u_1u_2 \ldots u_nu_0 \Rightarrow v_i = u_{i+1} \mod n \). For every \( u_i = 1 \), \( 0 \leq i \leq n-1 \), there exists \( x_i \) such that \( S = \{2^0, 2^1, 2^2, \ldots , 2^n\} \) and \( f(u) = S \). Since \( v_i = u_{i+1} \mod n \), there exists \( y_i, y_{i+1}, \ldots , y_k \) such that \( y_i = (x_i + 1) \mod n \) for all \( 1 \leq i \leq k \) and \( f(v) = S' = \{2^1, 2^2, \ldots , 2^n\} \).

Hence \( f(e) = f(uv) = f(u)f(v) = SS' \) is a shuffle edge in \( G^* \).

Conversely, let \( SS' \) be a shuffle edge in \( G^* \). That is, if \( S = \{2^0, 2^1, 2^2, \ldots , 2^n\} \), then \( S' = \{2^1, 2^2, \ldots , 2^n\} \) such that \( y_i = (x_i + 1) \mod n \) for all \( 1 \leq i \leq k \). This means \( y_i \) is moved one step forward from the position of \( x_i \) and if \( x_i = n - 1 \), then \( y_i = 0 \).

There exists \( u, v \in G \) such that \( u \) is a left cyclic shift of \( v \). Hence \( uv \) is a shuffle edge in \( G \).

Thus \( f \) preserves adjacency. \( \square \)

Remark 1- We draw this graph excluding the loops and parallel edges so that the graph is simple. See Figure 3.

Theorem 2 Let \( G \) be the shuffle-exchange network \( SE(n) \).

\[ |V(G)| = 2^n \quad \text{and} \quad |E(G)| = \begin{cases} 3 \times 2^{n-3} & \text{if } n \mod 2 \not= 0 \\ 3 \times 2^{n-2} & \text{otherwise} \end{cases} \]

Proof: Obviously \( |V(G)| = 2^n \). Degree of each vertex is 3. Hence there are \( 3 \times 2^{n-3} \) edges. Since the new drawing of \( SE(n) \) does not contain loops and parallel edges, by removing the loops, the number of edges becomes \( 3 \times 2^{n-3} - 2 \). In particular, the graphs \( SE(4) \) contain a pair of parallel edges in the middle row of vertices. Hence removing a parallel edge reduces the number of edges by one. Hence the result follows. \( \square \)

\( SE(6), SE(10), SE(14), \ldots \), which satisfy the condition
\[ |C|_{\gamma'} = \left\lfloor \frac{|C|_{\gamma'}}{n} \right\rfloor \times n = 2 \]

and the condition in Figure 3 The 4-dimensional Shuffle Exchange Network.

Figure 4 Shuffle-Exchange Network of dimension 4

Dotted lines represent deleted loops and parallel edge

Definition- The removal of the exchange edges partitions the graph into a set of connected components called necklaces. Each necklace is a ring of nodes connected by shuffle edges. Figure 4 shows shuffle exchange network \( SE(4) \). Figure 5 shows the necklaces of \( SE(4) \) after removing all the exchange edges. The nodes 0010, 0100, 1000 and 0001 form a 4-node necklace.

Figure 5 The necklaces of \( SE(4) \)

Theorem 3 The diameter of \( SE(n) \) is equal to \( 2n - 1 \).

Proof: The longest path between any two vertices of \( SE(n) \) is the path between the pendant vertices at the beginning and the end. Hence \( \text{diam}(SE(n)) = 2n - 1 \). \( \square \)
Conjecture 1 $SE(n)$, $n \geq 4$, does not contain a Hamiltonian path.
\[ \square \]

Theorem 4 The number of $n$-node necklaces in $SE(n)$ is
\[ \sum_{r=1}^{n-1} \frac{n}{n}. \]

Proof. The new drawing of $SE(n)$ shows that the number of vertices is divided into $n+1$ partitions each containing $\frac{n}{n}$, $\chi(C_{r+1}) = \chi(C_{n}) = n$.
\[ \square \]

Lemma 1 [4] Let $C_{p}$ denotes a $p$-cycle. If $\chi(C_{p}) = n$ and $k \geq 2$, then $\chi(C_{p+k}) \geq n$.
\[ \square \]

Lemma 2[4] If $\chi(C_{p}) = n$, then $p \geq \frac{n(n-1)}{2}$. Then
\[ \square \]

Lemma 3 [4] For $n \geq 2$ and even, let $p = \frac{n^{2}}{2}$. Then
\[ \chi(C_{p}) = n \text{ and } \chi(C_{p+1}) = n-1. \]
\[ \square \]

Lemma 4 [4] For $n \geq 3$ and odd, let $p = \frac{n(n-1)}{2}$.

In Figure 5, there are three $4$-node necklaces.

A. The Achromatic Labeling

Definition-The achromatic number of a graph $G$, denoted by $\chi(G)$, is the greatest number of colours in a vertex colouring such that for each pair of colours, there is at least one edge whose endpoints have those colours. Such a colouring is called a complete colouring. More precisely, the achromatic number for a graph $G = (V, E)$ is the largest integer $m$ such that there is a partition of $V$ into disjoint independent sets $(V_{1}, \ldots, V_{m})$ such that for each pair of distinct sets $V_{i}, V_{j}, V_{i} \cup V_{j}$ is not an independent set in $G$. The achromatic number of a path of length 4, $\chi(P_{4}) = 3$. See Figure 6. Yannakakis and Gavril proved that determining this value for general graphs is $NP$-complete [10].

Figure 6 Achromatic number of a path of length 4 is 3

The $NP$-completeness of the achromatic number problem holds also for some special classes of graphs: bipartite graphs [3], complements of bipartite graphs [8], cographs and interval graphs [1] and even for trees [7]. For complements of trees, the achromatic number can be computed in polynomial time [10]. The achromatic number of an $n$-dimensional hypercube graph is known to be proportional to $\sqrt{2^n}$, but the constant of proportionality is not known precisely [9]. In this section, we give an approximation algorithm to determine the achromatic number of shuffle-exchange network.

III. AN APPROXIMATION ALGORITHM FOR THE ACHROMATIC NUMBER OF $SE(n)$

Some of the basic results on achromatic number, which are in the literature, are given below.
edges. Then \( SE(n) \) has a complete \( k \) - colouring satisfying \[ k - \frac{1}{3} \leq 2^k. \] That is, \( k^2 - k - 3.2^n \leq 0 \) which implies \[ k \leq 1 + \sqrt{1 + 3.2^n}. \]

Thus \( \chi(SE(n)) \leq \frac{1 + \sqrt{1 + 3.2^n}}{2} \) which completes the proof. \( \Box \)

We have also derived a lower bound for the achromatic number of \( SE(n) \), when \( n \) is prime.

**Theorem 9** \( \chi(SE(n)) \geq \left( \sqrt{2^{n+1}} - 1 \right) \), when \( n \) is prime.

**Proof.** When \( n \) is prime, the number of \( n \) - node necklaces is given by \( T = \sum_{i=1}^{n-1} \frac{n!}{n} = \frac{1}{n} (2^n - 2) \). Let \( N = nT \). Let \( k \) be an integer such that \( N < (k + 1) \left( \frac{k}{2} \right) < \frac{(k + 1)^2}{2} \), which implies \( k^2 + 2N \). Therefore \( k > -1 + \sqrt{2N} \). In other words, \( k > \sqrt{2^{n+1}} - 1 \). Hence the theorem is proved. \( \Box \)

From Theorem 8 and Theorem 9, we obtain the following result.

**Theorem 10** There exists an \( O(1) \) - approximation algorithm to determine the achromatic number of Shuffle-Exchange network \( SE(n) \), when \( n \) is prime. \( \Box \)

**Conjecture 2** Theorem 10 holds for \( SE(n) \), for any \( n \). \( \Box \)

## IV. CONCLUSION

In this paper, we give a new method of drawing Shuffle Exchange Network of any dimension. Since the drawing of shuffle exchange network is complicated, many of its properties are hard to understand. The new representation which we have introduced helps us to study some of its properties. An approximation algorithm has been given for achromatic number of hypercubes \[9\]. In this paper, we have considered this problem for the Shuffle Exchange Network of prime dimension. The conjecture mentioned in this paper and the problem for deBrujin and torus are under investigation.

## REFERENCES


