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Shuffle Exchange Networks and Achromatic Labeling

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Abstract - Design of interconnection networks is an important integral part of the parallel processing or distributed systems. There are a large number of topological choices for interconnection networks. Among several choices, the Shuffle Exchange Network is one of the most popular versatile and efficient topological structures of interconnection networks. In this paper, we have given a new method of drawing shuffle exchange network for any dimension. This has enabled us to investigate some of the topological properties of shuffle-exchange network. Also we give an approximation algorithm for achromatic number of shuffle-exchange network.

Keywords- Interconnection networks, Shuffle Exchange Network, Achromatic Labeling.

I. INTRODUCTION

An interconnection network consists of a set of processors, each with a local memory, and a set of bidirectional links that serve for the exchange of data between processors. A convenient representation of an interconnection network is by an undirected (in some cases directed) graph $G = (V, E)$ where each processor is a vertex in V and two vertices are connected by an edge if and only if there is a direct (bidirectional for undirected and unidirectional for directed graphs) communication link between processors. We will use the term interconnection network and graph interchangeably.

A. The Shuffle Exchange Network

Definition -Let Q_n denote an n - dimensional hypercube. The n dimensional shuffle - exchange network, denoted by $SE(n)$, has vertex set $V = V(Q_n)$, and two vertices $x = x_1x_2\ldots x_n$ and $y = y_1y_2\ldots y_n$ are adjacent if and only if either

(i) x and y differ in precisely the n^{th} bit, or
(ii) x is a left or right cyclic shift of y.

The edge defined by the condition (i) is called an *exchange edge*, and (ii) is called a *shuffle edge*. The condition (ii) means that either $y_1y_2\ldots y_n = x_2x_3\ldots x_nx_1$ or $y_1y_2\ldots y_n = x_nx_1x_2\ldots x_{n-2}x_{n-1}$.

As an example, the 8-node shuffle exchange graph is given in Figure 1. The shuffle edges are drawn with solid lines while the exchange edges are drawn with dashed lines [6].For higher dimension, it is generally understood that drawing shuffle-exchange network is quite challenging.

We have redrawn the Shuffle-Exchange network considering the power set of $X = \{2^0, 2^1,\ldots, 2^{n-1}\}$ as vertices. This has enabled us to present a good drawing of the Shuffle-Exchange network for any dimension.

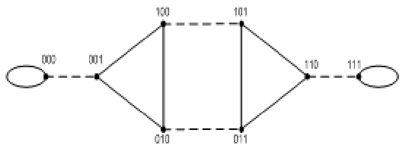

Figure 1 The 8 - node Shuffle Exchange Network

II. NEW REPRESENTATION OF THE SHUFFLE EXCHANGE NETWORK

Definition-Let $X = \{2^0, 2^1,\ldots, 2^{n-1}\}$ and let $P(X)$ denote the power set of X. We construct a graph G^* with vertex set $P(X)$ where
(i) Two nodes S and S' are adjacent if and only if $S \Delta S' = \{2^0\}$
(ii) If $|S| = |S'| = k$, where $S = \{2^{x_1}, 2^{x_2}, \ldots, 2^{x_k}\}$ and $S' = \{2^{y_1}, 2^{y_2}, \ldots, 2^{y_k}\}$, then S and S' are adjacent if and only if $y_i = (x_i + 1) \bmod n$ for all $1 \leq i \leq k$. See Figure 2.

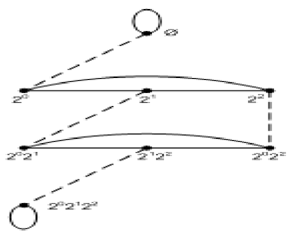

Figure 2 New Drawing of the Shuffle-Exchange Network

Theorem 1 The two definitions of Shuffle-Exchange network of dimension n are equivalent. In other words, the graph G^* constructed in definition 2 is isomorphic to the graph G given in definition 1.

Proof. Let G and G^* be the graphs defined by definition 1 and definition 2 respectively. First we associate with each binary string of length n, a set S in $P(X)$..

Let $u = (u_1, u_2, \ldots, u_n)$, $u_i \in \{0, 1\}$ be an arbitrary string. If $u_i = 1$, then let $2^{n-i} \in S$. If $u_i = 0$, then $2^{n-i} \notin S$. If $u_i = 0$ for every i, then $S = \varnothing$. For example, the string 000 is associated with $\varnothing$, 001 is associated with $\{2^0\}$, 010 with $\{2^1\}$, 100 with $\{2^2\}$, 011 with $\{2^0, 2^1\}$, 101 with $\{2^0, 2^2\}$, 110 with $\{2^1, 2^2\}$, and 111 with $\{2^0, 2^1, 2^2\}$.

Define $f : V(G) \rightarrow V(G^*)$ by $f(u) = f(u_1, u_2, \ldots, u_n) = S$ where $2^{n-i} \in S$ if $u_i = 1$, $2^{n-i} \notin S$ if $u_i = 0$, $1 \leq i \leq n$.
Clearly f is well-defined, for, $u = v \Rightarrow u_i = v_i, \forall\ i = 1, 2, \ldots, n$

$\Rightarrow f(u) = f(v)$
$\Rightarrow S = S'$

f is one - one: Let $S, S' \in V(G^*)$ such that $S = S' = \{2^{x_1}, 2^{x_2}, ..., 2^{x_k}\} \Rightarrow f(u) = f(u')$

$\Rightarrow u_i = 1 = u_i', 1 \leq i \leq k$
$\Rightarrow u = u'$.

f is onto: For every $S \in V(G^*)$, there exists $u = (u_1, u_2, ..., u_n) \in V(G)$ such that $u_i = 1$ if $2^{n-i} \in S$ and $u_i = 0$ if $2^{n-i} \notin S$, $1 \leq i \leq n$ satisfying $f(u) = S$.

f preserves adjacency: Let $e = uv$ be an exchange edge in *G*. To prove, $f(e) = SS'$ is an exchange edge in G^*.
By definition, *u* and *v* differ in exactly the n^{th} bit. That is, if $u_n = 0$, then $v_n = 1$ and vice versa $\Rightarrow 2^0 \in S$ and $2^0 \notin S'$, that is, if $S = \{2^0, 2^1, 2^2, ..., 2^{n-1}\}$, then $S' = \{2^1, 2^2, ..., 2^{n-1}\}$.

Hence $S \Delta S' = \{2^0\} \Rightarrow SS'$ is an exchange edge in G^*.
Conversely, Let SS' be an exchange edge in G^*. In other words, if $S = \{2^0, 2^{x_1}, 2^{x_2}, ..., 2^{x_k}\}$, then $S' = \{2^{x_1}, 2^{x_2}, ..., 2^{x_k}\}$.
This implies $u_{n-x_1} = u_{n-x_2} = ... = u_{n-x_k} = 1 = v_{n-x_1} = v_{n-x_2} = ... = v_{n-x_k}$, the rest of *u* and *v* are zero with $u_n = 1$ and $v_n = 0$.
Hence *u* and *v* differ in exactly the n^{th} bit.
$\Rightarrow uv$ is an exchange edge in *G*.

Let $e = uv$ be a shuffle edge in *G*. Then $u = (u_0, u_1, ..., u_{n-1})$ is a left (or a right) cyclic shift of $v = (v_0, v_1, ..., v_{n-1})$, that is, $v_0v_1 ...v_{n-1} = u_1u_2...u_{n-1}u_0 \Rightarrow v_i = u_{(i+1) \bmod n}$. For every $u_i = 1$, $0 \leq i \leq n-1$, there exists x_j such that $S = \{2^{x_1}, 2^{x_2}, ..., 2^{x_k}\}$ and $f(u) = S$. Since $v_i = u_{(i+1) \bmod n}$, there exists $y_1, y_2, ..., y_k$ such that $y_i = (x_i + 1) \bmod n$ for all $1 \leq i \leq k$ and $f(v) = S' = \{2^{y_1}, 2^{y_2}, ..., 2^{y_k}\}$.
Hence $f(e) = f(uv) = f(u)f(v) = SS'$ is a shuffle edge in G*
.
Conversely, let SS' be a shuffle edge in G*. That is, if $S = \{2^{x_1}, 2^{x_2}, ..., 2^{x_k}\}$, then $S' = \{2^{y_1}, 2^{y_2}, ..., 2^{y_k}\}$ such that $y_i = (x_i + 1) \bmod n$ for all $1 \leq i \leq k$. This means y_i is moved one step forward from the position of x_i and if $x_i = n - 1$, then $y_i = 0$. There exists $u, v \in G$ such that *u* is a left cyclic shift of *v*. Hence *uv* is a shuffle edge in *G*.
Thus *f* preserves adjacency. □

Remark 1- We draw this graph excluding the loops and parallel edges so that the graph is simple. See Figure 3.
Theorem 2 Let *G* be the shuffle-exchange network *SE*(*n*). Then $|V(G)| = 2^n$ and

$$|E(G)| = \begin{cases} 3 \times 2^{n-1} - 3 & \text{if } {}^nC_{\lceil n/2 \rceil} - \left\lfloor \frac{{}^nC_{\lceil n/2 \rceil}}{n} \right\rfloor \times n = 2 \\ 3 \times 2^{n-1} - 2 & \text{otherwise} \end{cases}.$$

Proof: Obviously $|V(G)| = 2^n$. Degree of each vertex is 3. Hence there are 3. 2^{n-1} edges. Since the new drawing of *SE*(*n*) does not contain loops and parallel edges, by removing the loops, the number of edges becomes 3. 2^{n-1} - 2. In particular, the graphs *SE*(4), *SE*(6), *SE*(10), *SE*(14), *etc* which satisfy the condition ${}^nC_{\lceil n/2 \rceil} - \left\lfloor \frac{{}^nC_{\lceil n/2 \rceil}}{n} \right\rfloor \times n = 2$ contain a pair of parallel edges in the middle row of vertices. Hence removing a parallel edge reduces the number of edges by one. Hence the result follows. □

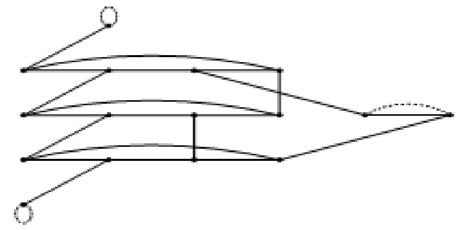

Figure 3 The 4-dimensional Shuffle Exchange Network.

Dotted lines represent deleted loops and parallel edge

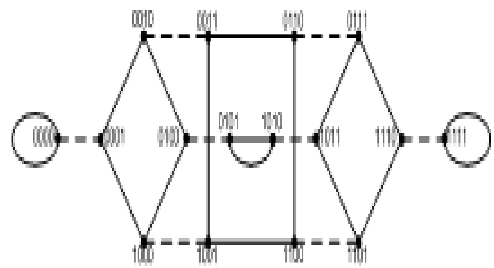

Figure 4 Shuffle-Exchange Network of dimension 4

Definition- The removal of the exchange edges partitions the graph into a set of connected components called *necklaces*. Each necklace is a ring of nodes connected by shuffle edges. Figure 4 shows shuffle exchange network *SE*(4). Figure 5 shows the necklaces of *SE*(4) after removing all the exchange edges. The nodes 0010, 0100, 1000 and 0001 form a 4-node necklace.

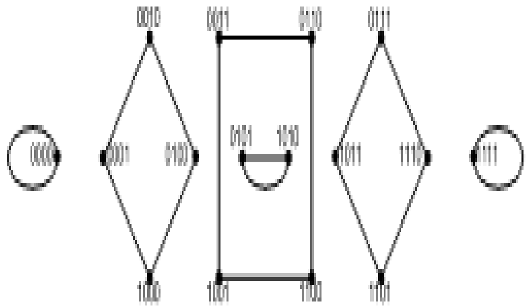

Figure 5 The necklaces of SE(4)

Theorem 3 The diameter of *SE*(*n*) is equal to $2n - 1$.

Proof. The longest path between any two vertices of *SE*(*n*) is the path between the pendant vertices at the beginning and the end. Hence *diam*(*SE*(*n*)) = 2*n* - 1. □

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Conjecture 1 $SE(n)$, $n \geq 4$, does not contain a Hamiltonian path. □

Theorem 4 The number of n - node necklaces in $SE(n)$ is $\sum_{r=1}^{n-1} \left\lfloor \frac{{}^{n}C_r}{n} \right\rfloor$.

Proof. The new drawing of $SE(n)$ shows that the number of vertices are divided into $n + 1$ partitions each containing ${}^{n}C_r$, $r = 0,1,2,\ldots,n$ vertices. Each partition contains $\left\lfloor \frac{{}^{n}C_r}{n} \right\rfloor$ number of n - node necklaces. Thus the number of n - node necklaces in $SE(n)$ is $\sum_{r=1}^{n-1} \left\lfloor \frac{{}^{n}C_r}{n} \right\rfloor$.□

In Figure 5, there are three 4 - node necklaces.

A. The Achromatic Labeling

Graph labeling enjoys many practical applications as well as theoretical challenges. Besides the classical types of problems, different limitations can also be set on the graph, or on the way a colour is assigned, or even on the colour itself. Scheduling, Register allocation in compilers, Bandwidth allocation, and pattern matching are some of the applications of graph colouring.

Definition-The achromatic number of a graph G, denoted by $\chi(G)$, is the greatest number of colours in a vertex colouring such that for each pair of colours, there is at least one edge whose endpoints have those colours. Such a colouring is called a complete colouring. More precisely, the achromatic number for a graph $G = (V, E)$ is the largest integer m such that there is a partition of V into disjoint independent sets $(V_1,\ldots,V_m)$ such that for each pair of distinct sets V_i, V_j, $V_i \cup V_j$ is not an independent set in G.The achromatic number of a path of length 4, $\chi(P_4) = 3$. See Figure 6. Yannakakis and Gavril proved that determining this value for general graphs is NP-complete [10].

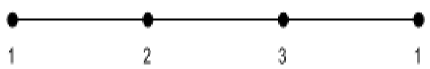

Figure 6 Achromatic number of a path of length 4 is 3

The NP-completeness of the achromatic number problem holds also for some special classes of graphs: bipartite graphs [3], complements of bipartite graphs [8], cographs and interval graphs [1] and even for trees [7]. For complements of trees, the achromatic number can be computed in polynomial time [10]. The achromatic number of an n-dimensional hypercube graph is known to be proportional to $\sqrt{n2^n}$, but the constant of proportionality is not known precisely [9].In this section, we give an approximation algorithm to determine the achromatic number of shuffle-exchange network.

III. AN APPROXIMATION ALGORITHM FOR THE ACHROMATIC NUMBER OF *SE*(*N*)

Some of the basic results on achromatic number, which are in the literature, are given below.

Lemma 1 [4] Let C_p denotes a p-cycle. If $\chi(C_p) = n$ and $k \geq 2$, then $\chi(C_{p+k}) \geq n$.□

Lemma 2[4] If $\chi(C_p) = n$, then $p \geq \frac{n(n-1)}{2}$.□

Lemma 3 [4] For $n > 2$ and even, let $p = \frac{n^2}{2}$. Then $\chi(C_p) = \chi(C_{p+1}) = n$.□

Lemma 4 [4] For $n \geq 3$ and odd, let $p = \frac{n(n-1)}{2}$. Then $\chi(C_p) = n$ and $\chi(C_{p+1}) = n-1$.□

Lemma 5 [4] If $p = \frac{n(n-1)}{2}$, $n \geq 3$, then $\chi(C_p) = n$. For p between $\frac{n(n-1)}{2}$ and $\frac{(n+1)n}{2}$, $\chi(C_p) = n$ unless n is odd and $p = \frac{n(n-1)}{2} + 1$, in which case $\chi(C_{p+1}) = n-1$.□

Lemma 6 [2] If C is an n-cycle with achromatic number $\chi(C)$,

then $n \geq \begin{cases} \frac{\chi(C)(\chi(C)-1)}{2}, & \text{if } \chi(C) \text{ is odd} \\ \frac{(\chi(C))^2}{2}, & \text{if } \chi(C) \text{ is even} \end{cases}$.□

Theorem 5 [5] Let $l_1, l_2, \ldots, l_k$ $(l_i \geq 3)$ be positive integers. Then there exists positive integers $l_1', l_2', \ldots, l_{k-1}'$ such that $\sum_{i=1}^{k-1} l_i' = \sum_{i=1}^{k} l_i$ and $\chi(C_{l_1} \cup C_{l_2} \cup \ldots \cup C_{l_k}) \leq \chi(C_{l_1'} \cup C_{l_2'} \cup \ldots \cup C_{l_{k-1}'})$. In particular, if $\sum_{i=1}^{k} l_i = p$, then $\chi(C_{l_1} \cup C_{l_2} \cup \ldots \cup C_{l_k}) \leq \chi(C_p)$, where C_t denotes a cycle of length t. □

Theorem 6 [5] Let $3 \leq l_1 \leq l_2 \leq \ldots \leq l_k$ be positive integers and let $\sum_{i=1}^{k} l_i = p$. If $k \leq \sqrt{\frac{p}{2}}$, then $\chi(C_{l_1} \cup C_{l_2} \cup \ldots \cup C_{l_k}) = \chi(C_p)$.□

Theorem 7 [5] Let G be a regular graph of degree m. If G has a complete n - colouring, then $\left\lceil \frac{n-1}{m} \right\rceil n \leq |V(G)|$.

The following is our result on the achromatic number of $SE(n)$.

Theorem 8 An upper bound for the achromatic number of $SE(n)$ is given by $\chi(SE(n)) \leq \frac{1+\sqrt{1+\left(3.2^{n+2}-16\right)}}{2}$.Proof. $SE(n)$ contains $T = \sum_{r=1}^{n-1} \left\lfloor \frac{{}^{n}C_r}{n} \right\rfloor$ number of n - node necklaces. We know that $SE(n)$ is a regular graph of degree 3 except the pendant vertices at the beginning and the end and the vertices containing the parallel edges. Then *SE*(*n*) has a complete *k* - colouring satisfying $\left\lceil \frac{k-1}{3} \right\rceil k \le 2^n$. That is, $k^2 - k - 3.2^n \le 0$ which implies $k \le \frac{1+\sqrt{1+4.3.2^n}}{2} \le \frac{1+\sqrt{1+3.2^{n+2}}}{2}$.

Thus $\chi(SE(n)) \le \frac{1+\sqrt{1+\left(3.2^{n+2}-16\right)}}{2}$ which completes the proof. □

We have also derived a lower bound for the achromatic number of *SE*(*n*), when *n* is prime.

Theorem 9 $\chi(SE(n)) \ge \left(\sqrt{2^{n+1}-4}\right)-1$, when *n* is prime.

Proof. When *n* is prime, the number of *n* - node necklaces is given by $T = \sum_{r=1}^{n-1} \frac{^nC_r}{n} = \frac{1}{n}\left(2^n - 2\right)$. Let $N = nT$. Let *k* be an integer such that $N < (k+1)\binom{k}{2} < \frac{(k+1)^2}{2}$, which implies $(k+1)^2 > 2N$.

Therefore $k > -1 + \sqrt{2N}$. In other words, $k > \sqrt{2^{n+1}-4} - 1$. Hence the theorem is proved. □

From Theorem 8 and Theorem 9, we obtain the following result.

Theorem 10 There exists an *O*(1) - approximation algorithm to determine the achromatic number of Shuffle-Exchange network *SE*(*n*), when *n* is prime. □

Conjecture 2 Theorem 10 holds for *SE*(*n*), for any *n*. □

IV. CONCLUSION

In this paper, we give a new method of drawing Shuffle Exchange Network of any dimension. Since the drawing of shuffle exchange network is complicated, many of its properties are hard to understand. The new representation which we have introduced helps us to study some of its properties.An approximation algorithm has been given for achromatic number of hypercubes [9]. In this paper, we have considered this problem for the Shuffle Exchange Network of prime dimension. The conjecture mentioned in this paper and the problem for deBrujin and torus are under investigation.

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