Graceful Labeling of Bow Graphs and Shell-Flower Graphs

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Abstract - A graceful labeling of a graph G with 'q' edges and vertex set V is an injection f: V(G) → {0,1,2,...,q} with the property that the resulting edge labels are also distinct, where an edge incident with vertices u and v is assigned the label |f(u)−f(v)|. A graph which admits a graceful labeling is called a graceful graph. A Shell graph is defined as a cycle C_n with (n-3) chords sharing a common end point called the apex. Shell graphs are graceful for n = 4. A graph which admits a graceful labeling is called a graceful graph. A Shell graph is defined as a cycle C_n with (n-3) chords sharing a common end point called the apex. Shell graphs are graceful for n = 4.

Keywords: Graceful labeling, shell graph, Bow graph, shell-flower graph.

I. INTRODUCTION

In 1967 Rosa [11] introduced the labeling method called β - valuation as a tool for decomposing the complete graph into isomorphic sub graphs. Later on, this β - valuation was renamed as graceful labeling by Golomb [6]. A graceful labeling of a graph G with 'q' edges and vertex set V is an injection f : V(G) → {0,1,2,...,q} with the property that the resulting edge labels are also distinct, where an edge incident with vertices u and v is assigned the label |f(u)−f(v)|. A graph which admits a graceful labeling is called a graceful graph. Various kinds of graphs are shown to be graceful. In particular, cycle - related graphs have been a major focus of attention for nearly five decades. Rosa[11] showed that the n - cycle C_n is graceful if and only if n ≠ 0 or 3 (mod 4). Frucht [4] has shown that the Wheels W_n,C_n+ K_1 are graceful. Helmsh (graph obtained from a wheel by attaching a pendant edge at each vertex of the n - cycle) are shown to be graceful by Ayel and Favaron[1]. Koh, Rogers, Teo and Yap[9] defined a web graph as one obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. The web graph is proved to be graceful by Kang, Liang, Gao and Yang [8].

Delorme, Kohet al [3] showed that any cycle with achord is graceful. In 1985 Koh, Rogers, Teo and Yap[10] defined a cycle with a P_3 -chord to be a cycle with the path P_3 joining two non-consecutive vertices of the cycle and proved that these graphs are graceful when k = 3. For an exhaustive survey, refer to the dynamic survey by Gallian[5].

Deb and Limaye [2] have defined a shellgraph as a cycle C_n with (n-3) chords sharing a common end point called the apex. Shell graphs are denoted as C(n, n-3) (see Figure 1).

Note that the shell C(n, n-3) is the same as the fan F_n = P_n + K_1. A multiple shell is defined to be a collection of edge disjoint shells that have their apex in common. Hence a double shell consists of two edge disjoint shells with a common apex. A bow graph is defined to be a double shell in which each shell has any order. In this paper we prove that the bow graph with shell orders 'm' and '2m' is graceful. Further in this paper we define a shell - flower graph as 'k' copies of [C(n, n-3) U K_3] and we prove that all shell - flower graphs are graceful for n = 4.

In this paper we prove that all bow graphs with shells of order m and 2m excluding the apex are graceful. Further we define a shell-flower graph as 'k' copies of the union of the shell C(n, n-3) and K_2 where one end vertex of K_2 is joined to the apex of the shell. We denote this graph by [C(n, n-3) U K_3]^k where the superscript k denotes the k copies of [C(n, n-3) U K_3](see Figure 2) and we prove that all shell-flower graphs are graceful when n=4.

Figure 1. Shell graph C(n, n-3)

Figure 2. Shell-flower graph [C(n, n-3) U K_3]^k when n = 4
II. MAIN RESULT

In this section we first prove that bow graphs with shells of order \( m \) and \( 2m \) excluding the apex are graceful and secondly we prove all shell-flower graphs are graceful.

Theorem 1: All Bow graphs with shell orders ‘\( m \)’ and ‘\( 2m \)’(order excludes the apex) are graceful.

Proof: Let \( G \) be a bow graph with shells of order \( m \) and \( 2m \) excluding the apex. Let the number of vertices in \( G \) be \( n \) and the number of edges be \( q \). We describe the graph \( G \) as follows: In \( G \), the shell that is present to the left of the apex is called as the left wing and the shell that is present to the right of the apex is considered as the right wing. Let \( m \) be the order of the right wing of \( G \) and \( 2m \) be the order of the left wing of \( G \). The apex of the bow graph is denoted as \( v_0 \).

Denote the vertices in the right wing of the bow graph from bottom to top as \( v_{2m}, v_{2m+1}, v_{2m-1} \). The vertices in the left wing of the bow are denoted from top to bottom as \( v_0, v_1, v_2, \ldots, v_{3m} \). Note that \( n = (3m + 1) \) and \( q = (6m - 2) \). (See Figure 3).

![Figure 3. A Bow Graph with n = (3m + 1) vertices](image)

We label the vertices of the bow graph as follows.

**Case 1: When \( m \) is odd.**

Define

\[
\begin{align*}
\text{if } & (v_0) = 0, \quad f(v_{2i}) = 6m - 2i, \quad \text{for } 1 \leq i \leq (m + 1)/2 \\
\text{if } & (v_{2i+1}) = 4m + 2i - 2, \quad \text{for } 1 \leq i \leq (m + 1)/2 \\
\text{if } & (v_{2m}) = 3m + 2i - 2, \quad \text{for } (m + 1)/2 \leq i \leq (3m - 1)/2
\end{align*}
\]

From the above definition given in (1), (2), (3) we see that the vertices have distinct labels.

We compute the edge labels as follows.

\[
\begin{align*}
\text{if } & (v_0) - f(v_{2i}) = \left| 4m + 2i - 2, \text{for } 1 \leq i \leq (m + 1)/2 \right| \\
\text{if } & (v_{2i+1}) - f(v_{2i}) = \left| 2m - 4i + 2, \text{for } 1 \leq i \leq (m - 1)/2 \right| \\
\text{if } & (v_{2m}) - f(v_{2i+1}) = \left| 2m - 4i + 1, \text{for } (m + 1)/2 \leq i \leq (3m - 1)/2 \right|
\end{align*}
\]
Define
\[ f(v_0) = 0 \] (15)
\[ 3i - 1, \text{ for } i = 1, 3, 5, \ldots, (2k-1) \]
\[ f(v_{2i-1}) = 3i - 2, \text{ for } i = 2, 4, 6, \ldots, (2k) \] (16)
\[ 4m+i, \text{ for } i = 1, 3, 5, \ldots, (2k-1) \]
\[ f(v_2i) = 3i, \text{ for } i = 2, 4, 6, \ldots, (2k). \] (17)

From the above definition given in (15), (16) and (17) we see that the vertices have distinct labels.

We compute the edge labels as follows.

\[ |f(v_i) - f(v_{2i-1})| = 3i - 1, \text{ for } i = 1, 3, 5, \ldots, (2k-1) \]
\[ 3i - 2, \text{ for } i = 2, 4, 6, \ldots, (2k), \] (18)
\[ |f(v_i) - f(v_{2i})| = 4k + i, \text{ for } i = 1, 3, 5, \ldots, (2k-1) \]

From the computations given in (18), (19), (20), (21) we can see that the edge labels are distinct. Hence all shell flower graphs \([C(n, n-3) U K_k]\) are graceful when \(n=4\). The illustration for the shell-flower is given in the appendix.

### III. CONCLUSION

In this paper we have proved the gracefulfulness of the bow graph with shell orders ‘m’ and ‘2m’ and the shell-flower graphs \([C(n, n-3) U K_2]^k\) when \(n = 4\).

**Appendix**

Figure A. Graceful bow graph when \(m = 6\)

Figure B. Graceful bow graph when \(m = 5\)

![Figure C. Graceful shell-flower graph](image)

**REFERENCES**


